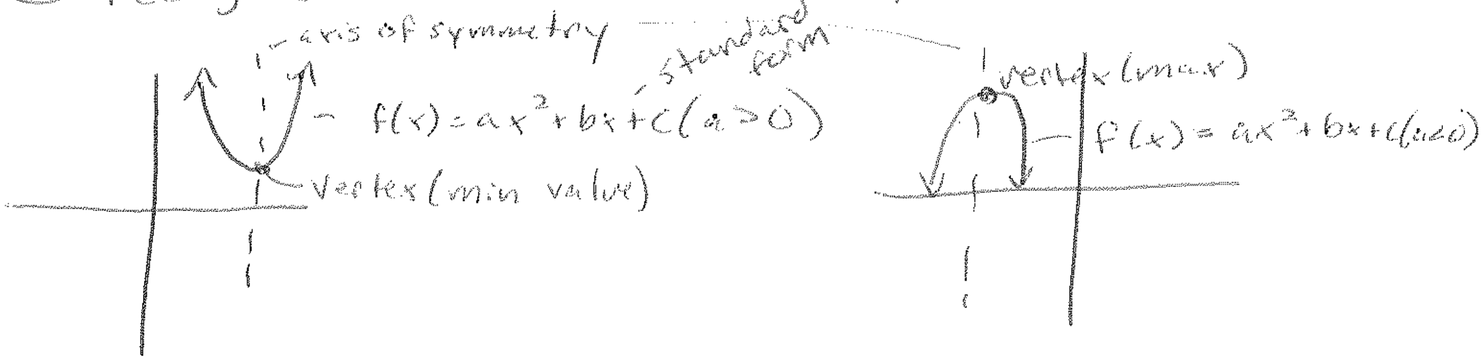
Chapter 2.2 Quadratic Functions

① Recognize characteristics of parabolas



② Graph parabolas

Standard or graphing form

$$f(x)=a(x-h)^2+k$$

Vertex: (opp h, keep k)
Negative a: reflect
a: stretch ($a>1$) comp ($0<a<1$)

Ex 1) Graph the quadratic function $f(x)=-(x-1)^2+4$
→ Also talk about intercepts

③ Graphing a Quadratic function in the form
$f(x)=ax^2+bx+c$

Ex 1) $f(x)=-x^2+4x+1$

$a=-1$
$b=4$
$c=1$

Step 1: Determine how the ~~graph~~ parabola opens,
Since $a<1$
∴ downwards

Step 2: Find the vertex $f(x) = -x^2 + 4x + 1$

$a = -1$

$b = 4$

$c = 1$

$$x = -\frac{b}{2a} = \frac{-4}{2(-1)} = \frac{-4}{-2} = 2$$

Now substitute x into f(x)

$f(2) = -(2)^2 + 4(2) + 1$

$= -4 + 8 + 1$

$= 5$

Vertex at $(2, 5)$ ⟵ Maximum

Step 3: Find the x-intercepts by solving $f(x) = 0$
↳ set $y = 0$

$0 = -x^2 + 4x + 1$ — try to factor first

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(-1)(1)}}{2(-1)} = \frac{-4 \pm \sqrt{16 + 4}}{-2} = \frac{-4 \pm \sqrt{20}}{-2}$$

$$x = \frac{-4 + \sqrt{20}}{-2} \approx \qquad x = \frac{-4 - \sqrt{20}}{-2} \approx$$

$(-0.24, 0)$ and $(4.24, 0)$

Step 4: Find the y-int by computing $f(0)$.

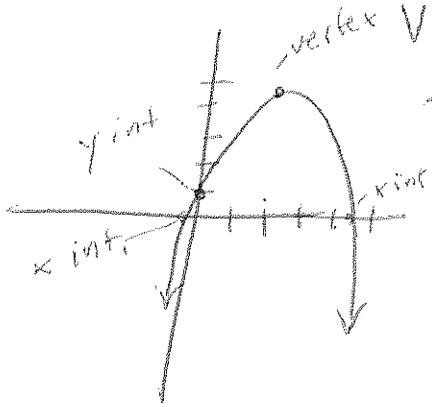
$f(0) = -(0)^2 + 4(0) + 1$

$f(0) = 1$

$\therefore$ y int. $= (0, 1)$

Step 5: Graph the parabola

Downwards



V: $(2, 5)$

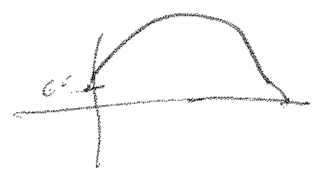
x int: $(-0.24, 0)$ & $(4.24, 0)$

y int: $(0, 1)$

D: $(-\infty, \infty)$

R: $(-\infty, 5]$

- Talk about football question
  - how long in air?
  - how far?
  - max height?



- (4) Among all pairs of numbers whose difference is 8, find a pair whose product is as small as possible. What is the minimum product?

- Let x = one of the numbers

  $x - 8$ = the other #.

- The product is $f(x) = x(x-8) = x^2 - 8x$

- The x-coordinate of the minimum is

$$x = \frac{-b}{2a} = \frac{-(-8)}{2(1)} = \frac{8}{2} = 4$$

$f(4) = (4)^2 - 8(4)$

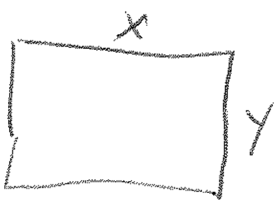
$= 16 - 32 = -16$

The vertex $= (4, -16)$

The min product $= -16$. This occurs when the two #s are 4 & $4 - 8 = -4$.

(5) You have 120 ft of fencing to enclose a rectangular area. Find the dimensions of the rect. that maximize the enclosed area. What is the max area?

- 120 ft of fencing.



$$2x+2y=120$$

$$\frac{2y}{2}=\frac{120-2x}{2}$$

$$y=60-x$$

We need to maximize $A=xy=x(60-x)$.

$$A=60x-x^2=-x^2+60x$$

- Since a is negative, opens downward, max. at

$$x=\frac{-b}{2a}=\frac{-60}{2(-1)}=\frac{-60}{-2}=30$$

So $x=30$ $\quad y=60-30=30$

$\therefore$ Max Area $=30\cdot 30=\underline{900\ ft^2}$

Chapter 2.3 Polynomial Functions & Their Graphs

① Identify polynomial functions

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

Where : $a_n \neq 0$

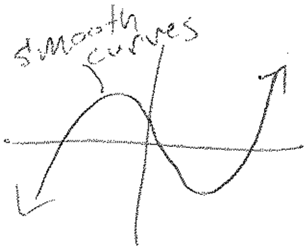
n is a nonnegative integer

note: $f(x) = c$ is a polynomial of degree 0

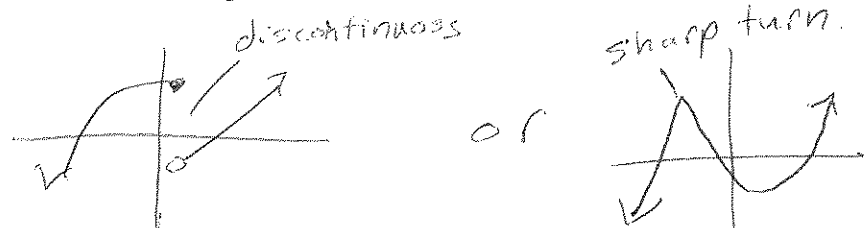
$f(x) = mx + b$ is " " 1

② Recognize characteristics of graphs of polynomial functions.

Yes



No



- with equations, not function if divided by x or with radical.

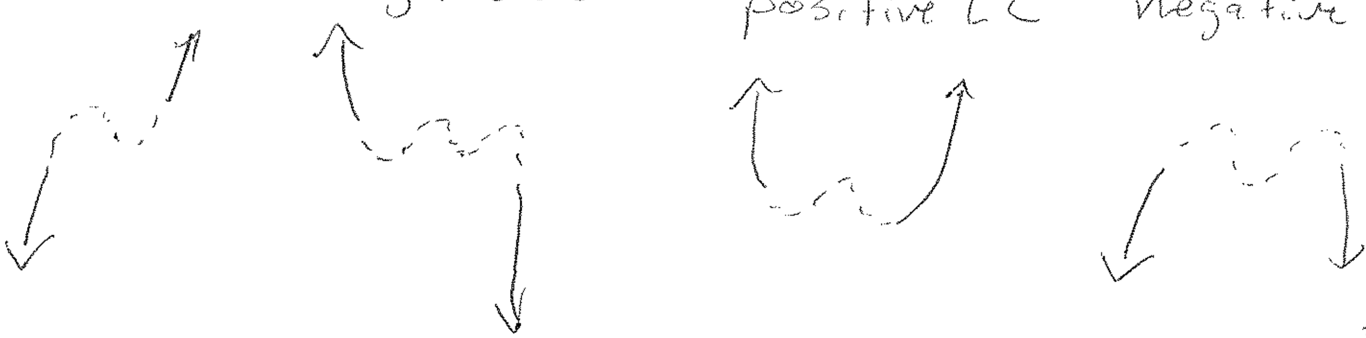
③ Determine End behavior.

For n odd:

positive LC negative LC

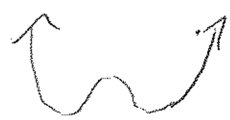
For n even:

positive LC negative LC



Ex 1 Use leading coef. test to determine End behavior

$f(x) = x^4 - 4x^2$



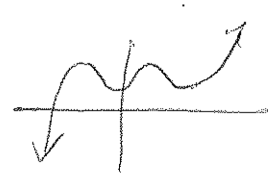
even degree
positive LC $\Rightarrow$

# of curves = 1 less than degree

Ex 2 $f(x) = 2x^3(x-1)(x+5)$

Degree = $3+1+1=5$

positive LC



(4) Use factoring to find zeros of polynomial functions

Ex 3 Find all zeros of $f(x) = x^3 + 2x^2 - 4x - 8$

$(x^3 + 2x^2)(-4x-8)$

$x^2(x+2) - 4(x+2$

$(x+2)(x^2-4)$

$(x+2)(x+2)(x-2) = 0$

$\therefore x = \pm 2$

Ex 4 $f(x) = x^4 - 4x^2$

$0 = x^2(x^2-4)$

$= x^2(x+2)(x-2)$

$x = 0, \pm 2$

⑤ Identify zeros and their multiplicities

Ex 5) $f(x) = -4(x+\frac{1}{2})^2(x-5)^3$

zeros: $x = -\frac{1}{2}$ with multiplicity of 2 — even touches

$x = 5$ with multiplicity of 3 — odd goes through

⑥ Use the Intermediate Value Theorem.

Ex 6) Show that the polynomial function $f(x) = 3x^3 - 10x + 9$ has a real zero between -3 & -2.

$f(-3) = 3(-3)^3 - 10(-3) + 9$
$= 3(-27) + 30 + 9$
$= -81 + 39 = -42$

$f(-2) = 3(-2)^3 - 10(-2) + 9$
$= 3(-8) + 20 + 9$
$= 5$

sign changes so ∴ there is a zero between -3 & -2.

⑦ Graph polynomial functions.

Ex 7) Use the 5 step strategy to graph $f(x) = x^3 - 3x^2$

Step 1: Determine the end behavior

odd degree
positive LC
$3 - 1 = 2$ turns

Step 2: Find x-intercepts by setting $f(x) = 0$

$x^3 - 3x^2 = 0$
$x^2(x-3) = 0$ $\quad x = 0, 3$ $\quad$ pts at $(0,0)$ & $(0,3)$

Step 3: Find the y-intercept by computing $f(0)$.

$f(0) = (0)^3 - 3(0)^2$

$f(0) = 0$ $\quad\therefore$ pt at $(0, 0)$

Step 4: Use possible symmetry to help draw the graphs.

$f(x) = x^3 - 3x^2$

$\downarrow$ Replace x w/ $-x$

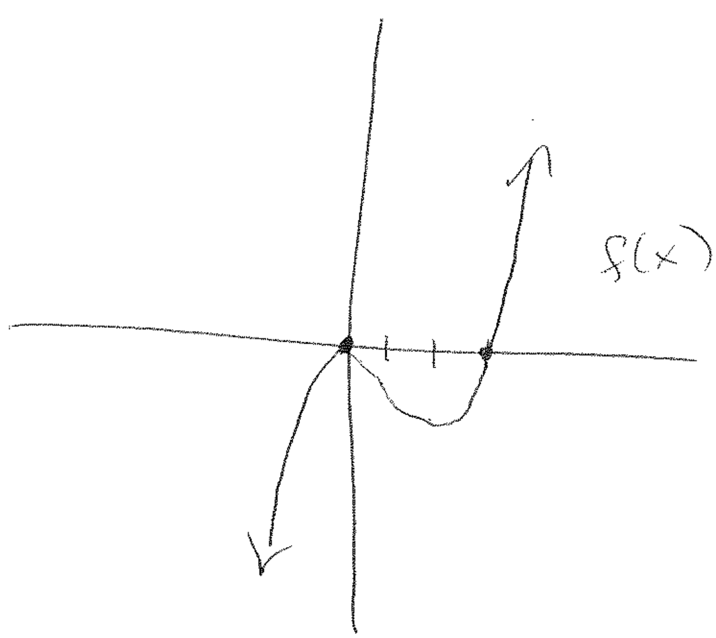
$f(-x) = (-x)^3 - 3(-x)^2$

$= -x^3 - 3x^2$

$\therefore$ No symmetry

Step 5: Turning pts?

2 turns (1 less than degree)



Chapter 2.4 Dividing Polynomials: Remainder and Factor theorms.

① Use long division to divide polynomials

Ex 1 Divide $x^2+14x+45$ by $x+9$

$$\begin{array}{r l}
 & x+5 \\
x+9 & \overline{)\,x^2+14x+45} \\
 & -\underline{x^2 + 9x} \\
 & 5x+45 \\
 & -\underline{5x + 45} \\
 & 0
\end{array}$$

$\boxed{x+5}$

Ex 2 Divide $7-11x-3x^2+2x^3$ by $x-3$

$$\boxed{2x^2+3x-2+\frac{1}{x-3}}$$

$$\begin{array}{r l}
x-3 & \overline{)\,2x^3-3x^2-11x+7} \\
 & -\underline{2x^3+6x^2} \\
 & 3x^2-11x+7 \\
 & -\underline{3x^2+9x} \\
 & -2x+7 \\
 & \underline{+2x-6} \\
 & 1
\end{array}$$

Ex 3 Divide $2x^4+3x^3-7x-10$ by x^2-2x

$$\begin{array}{r l}
 & 2x^2+7x+14 \\
x^2-2x & \overline{)\,2x^4+3x^3-7x-10} \\
 & -\underline{2x^4+4x^3} \\
 & 7x^3-7x-10 \\
 & -\underline{7x^3+14x^2}
\end{array}$$

$$\begin{array}{r l}
 & 14x^2-7x-10 \\
 & -\underline{14x^2+28x} \\
 & 21x-10
\end{array}$$

$$\boxed{\therefore 2x^2+7x+14+\frac{21x-10}{x^2-2x}}$$

(a) Use synthetic division

Ex 4) Use synthetic $\div$ to $\div$ $x^3 - 7x - 6$ by $x + 2$

missing an x^2

$x + 2 = 0$

$x = -2$

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -7 & -6 \\ & & -2 & 4 & 6 \\ \hline & 1 & -2 & -3 & \boxed{0} \end{array}$$

$$\boxed{x^2 - 2x - 3}$$

(3) Evaluate a polynomial using the Remainder Theorem.

Ex 5) Given $f(x) = 3x^3 + 4x^2 - 5x + 3$, use the remainder Theorem to find $f(-4)$.

$$\begin{array}{r|rrrr} -4 & 3 & 4 & -5 & 3 \\ & & -12 & 32 & -108 \\ \hline & 3 & -8 & 27 & \boxed{-105} \end{array}$$

$\therefore f(-4) = -105$

(4) Use the factor Theorem to solve a polynomial Eq.

Ex 6) Solve the eq. $15x^3 + 14x^2 - 3x - 2 = 0$ given that -1 is a zero

$$\begin{array}{r|rrrr} -1 & 15 & 14 & -3 & -2 \\ & & -15 & 1 & 2 \\ \hline & 15 & -1 & -2 & \boxed{0} \end{array}$$

$15x^2 - x - 2 = 0$

$(5x - 2)(3x + 1) = 0$

$\therefore x = \frac{2}{5}, -\frac{1}{3}$

The solution set is $\{-1, \frac{2}{5}, -\frac{1}{3}\}$

Ch. 2.5 Zeros of Polynomial Functions

① Use the Rational Zero Theorem to find possible rational zeros

P ÷ Qs

P => constant factors

Q => Leading Coefficient factors

$\frac{P}{Q}$ = potential zeros

Ex 1 List all possible rational zeros of

$f(x) = x^3 + 2x^2 - 5x - 6$

$P = 6 \Rightarrow \pm 1, \pm 2, \pm 3, \pm 6$

$Q = 1 \Rightarrow \pm 1$

$\boxed{\frac{P}{Q} = \pm 1, \pm 2, \pm 3, \pm 6}$

Ex 2 $f(x) = 4x^5 + 12x^4 - x - 3$

$P = 3 \Rightarrow \pm 1, \pm 3$

$Q = 4 \Rightarrow \pm 1, \pm 2, \pm 4$

$\frac{P}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}$

② Find zeros of a polynomial function

Ex 3 $f(x) = x^3 + 8x^2 + 11x - 20$

$P = 20 \Rightarrow \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

$q = 1 \Rightarrow \pm 1$

$\frac{P}{q} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

$$\begin{array}{r|rrrr} 1 & 1 & 8 & 11 & -20 \\ & & 1 & 9 & 20 \\ \hline & 1 & 9 & 20 & \boxed{0} \end{array}$$

$x^2 + 9x + 20$

$(x+4)(x+5) = 0$ ⇐ What happens if we can't factor?

$x = 1, -4, -5$

PS 1

Ex 4 Solve $x^4-6x^3+22x^2-30x+13=0$

$P=13 \Rightarrow \pm1, \pm13$

$Q=1 \Rightarrow \pm1$

$\frac{P}{Q}=\pm1, \pm13$

$$\begin{array}{r|rrrrr} 1 & 1 & -6 & 22 & -30 & 13 \\ & & 1 & -5 & 17 & -13 \\ \hline & 1 & -5 & 17 & -13 & \boxed{0} \end{array}$$

$(x-1)(x^3-5x^2+17x-13)$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 17 & -13 \\ & & 1 & -4 & 13 \\ \hline & 1 & -4 & 13 & \boxed{0} \end{array}$$

Not factorable

$(x-1)(x-1)(x^2-4x+13)$

$a=1$
$b=-4$
$c=13$

$$= \frac{4 \pm \sqrt{(-4)^2-4(1)(13)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16-52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

$$\boxed{\{1, 2+3i, 2-3i\}}$$

(4) Use the Linear Factorization Theorem to find polynomials with given zeros.

From last example: $(x-1)(x-1)(x-(2+3i))(x-(2-3i))$

↳ These are four zeros.

↳ These are linear factors

→ The Linear Factorization Theorem

$$f(x)=a_n x^n + a_{n-1}x^{n-1}+\dots+a_1 x + a_0$$

where: $n \geq 1$ & $a_n \neq 0$

then. $f(x)=a_n(x-c_1)(x-c_2)\dots(x-c_n)$

where $c_1, c_2, \dots, c_n$ are complex #s

Ex] Find a third-degree poly func. $f(x)$ w/real coef. that has -3 and i as zeros and such that $f(1)=8$ $\quad f(x)=a_n(x-c_1)(x-c_2)(x-c_3)$

$$(x+3)(x-i)(x+i)=(x+3)(x^2+1)$$

$$f(x)=a_n(x+3)(x^2+1)$$

$$f(1)=a_n(1+3)(1^2+1)$$

$$f(1)=a_n(4)(2)$$

$$f(1)=8a_n \qquad f(1)=8$$

$$\therefore a_n=1$$

so $f(x)=(x+3)(x^2+1)$ or x^3+3x^2+x+3

~~Ex~~ (5) Use Descartes's Rule of Signs LC2.5

1.) # of <u>positive</u> real zeros:

a.) the same number of sign changes of $f(x)$.

or

b.) Less than the # of sign changes of $f(x)$ by a positive even integer.

2.) # of <u>Negative</u> real zeros:

a.) " " $f(-x)$

or

b.) " " of $f(-x)$ " ".

Ex | Determine the possible numbers of positive and negative real zeros of $f\cancel{(x) = x^4 - 14x^3} + 71x^2$ $\cancel{+15}$

$f(x) = x^4 - 14x^3 + 71x^2 - 154x + 120$

sign change " " "

So possible positive real zeros $= 4, 2, 0$

$f(-x) = x^4 + 14x^3 + 71x^2 + 154x + 120$

Possible negative real zeros $= 0$. Since no sign change.

Chapter 2.6 Rational Functions and Their Graphs

① Find the domain of rational functions

Ex 1 a.) $f(x) = \frac{x^2-25}{x-5}$ b.) $g(x) = \frac{x}{x^2-25}$ c.) $\frac{x+5}{x^2+25}$

$D: \{x \mid x \neq 5\}$

$= \frac{x}{(x+5)(x-5)}$

$D: \{x \mid x \neq \pm 5\}$

$D: \{x \mid x \text{ is } \mathbb{R}\}$

② Use arrow notation

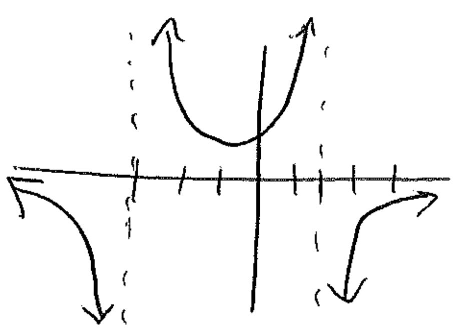
$x \to a^+$ x approaches a from the rt side

$x \to a^-$ x approaches a from the lt. side

$x \to \infty$ x approaches ∞

$x \to -\infty$ x " " $-\infty$

Ex 2



as $x \to -3^-$ $f(x) \to -\infty$

as $x \to -3^+$ $f(x) \to \infty$

③ Identify vertical Asymptotes

Ex 3 a.) $f(x) = \frac{x}{x^2-1}$

$f(x) = \frac{x}{(x-1)(x+1)}$

$x = \pm 1$

b.) $\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}} = \frac{1}{x+1}$

$x = -1$

c.) $h(x) = \frac{x-1}{x^2+1}$

No VA

(4) Identify Horizontal Asymptotes

- ~~Check~~ Make sure the degree matches in the numerator and denominator.
- Take leading Coef.

Ex 4 HA?

a.) $f(x) = \dfrac{9x^2}{3x^2+1}$

$HA \Rightarrow y = \frac{9}{3} = 3$

b.) $g(x) = \dfrac{0x^2+9x}{3x^2+1}$

$HA: y = \frac{0}{3} = 0$

c.) $h(x) = \dfrac{9x^3}{3x^2+1}$

$HA: y = \frac{9}{0} =$ und

None.

(5) Use transformations to graph rational functions

$$y = \frac{a}{x-h} + k \leftarrow \text{Graphing form}$$

If not in graphing form.

Symmetrical? $\longrightarrow$ $f(-x) = f(x)$ y-axis symmetry

$f(-x) = -f(x)$ origin symmetry

HA?

VA?

Factor & simplify

pick a point

Graph

Ex 5 $f(x) = \dfrac{2x-1}{x-1}$

$HA: y = \frac{2}{1} = 2$

$x-1=0$

$x=1$

Can't factor

$VA: x-1=0 \Rightarrow x=1$

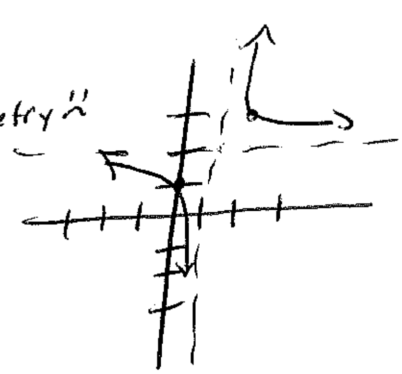
pick a pt.

if $x=0$ $y = \dfrac{2(0)-1}{0-1} = \dfrac{-1}{-1} = 1$ $(0,1)$

$f(-x) = \dfrac{2(-x)-1}{(-x)-1}$

$= \dfrac{-2x-1}{-x-1}$

No symmetry!!



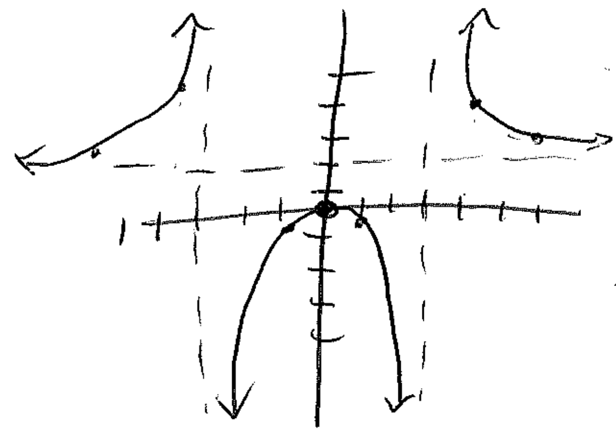
$\boxed{\text{Ex 6}}$ $f(x) = \dfrac{2x^2}{x^2-9}$

- Symmetry?

$f(-x) = \dfrac{2(-x)^2}{(-x)^2-9} = \dfrac{2x^2}{x^2-9} \Rightarrow$ y-axis symmetry

- HA?

$y = \frac{2}{1} = 2$



- VA?

factor $\dfrac{2x^2}{(x+3)(x-3)}$

$x = \pm 3$

- Pick points

$x = 0$

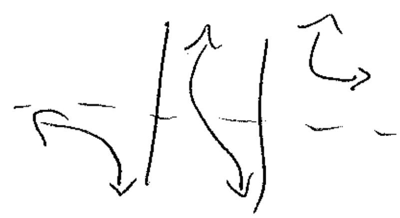
$y = \dfrac{2(0)^2}{0^2-9} = 0$

$x = 1$

$y = \dfrac{2(1)^2}{1^2-9} = \dfrac{2}{-8} = -\frac{1}{4}$

$x = 4$

$y = \dfrac{2(4)^2}{4^2-9} = \dfrac{32}{7} \approx 4.6$



Ex 7) $f(x) = \dfrac{x^4}{x^2+2}$

no HA

~~Ex 8~~ (6) Slant Asymptotes

-> Slant asymptote if degree of numerator is 1 more than the denominator

-> slant asymptote found using synthetic division

Ex 8) $F(x) = \dfrac{2x^2-5x+7}{x-2}$

$$\begin{array}{r|rrr} 2 & 2 & -5 & 7 \\ & & 4 & -2 \\ \hline & 2 & -1 & \boxed{5} \end{array} \qquad \underbrace{2x-1}_{} + \frac{5}{x-2}$$

Equation of slant asymptote

VA: $x = 2$

Pick a point

$x = 0$

$y = \dfrac{2(0)^2-5(0)+7}{0-2} = -\dfrac{7}{2} = -3.5$