Lesson 9.4 Counting Principles, Permutations, and Combinations

1. If you roll a die and flip a coin, how many possible outcomes can there be?
   a. How many possible outcomes are there when you roll a die?

      6

   b. How many possible outcomes are there when you flip a coin?

      2 (H, T)

   c. List all the possible outcomes of rolling a die and flipping a coin below:

      1H 2H 3H 4H 5H 6H
      1T 2T 3T 4T 5T 6T

      12 possible outcomes
      → $6 \cdot 2$

   d. Make a conjecture: If one event has m possible outcomes and another event has n possible outcomes, the number of possible outcomes if the events occur successively is $m \cdot n$.

2. For a college interview, Robert has to choose what to wear from the following: 2 slacks, 3 shirts, and 2 shoes. How many possible outfits does he have to choose from?

   $2 \cdot 3 \cdot 2 = 12$ outfits

The Fundamental Counting Principal

The number of ways in which a series of successive things can occur is found by multiplying the number of ways in which each thing can occur.

Example 1: Next semester you are planning to take three courses—math, English, and humanities. Based on time blocks and highly recommended professors, there are 8 sections of math, 5 of English, and 4 of humanities that you fi nd suitable. Assuming no scheduling conflicts, how many different three-course schedules are possible?

$8 \cdot 5 \cdot 4 = 160$ possible schedules

Example 2: Telephone numbers in the United States begin with three-digit area codes followed by seven-digit local telephone numbers. Area codes and local telephone numbers cannot begin with 0 or 1. How many different telephone numbers are possible?

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$8 \cdot 10 \cdot 10 \cdot 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

10 possible digits:
0, 1, 2, 3, 4, 5, 6, 7, 8, 9
(without 0 or 1 → only 8)

$= 64 \times 10^8$
$= 6{,}400{,}000{,}000$
possible phone numbers

Exploration 1: You are the coach of a little league baseball team. There are 13 players on the team (and lots of parents hovering in the background, dreaming of stardom for their little "Albert Pujols"). You need to choose a batting order having 9 players. The order makes a difference, because, for instance, if bases are loaded and "Little Albert" is fourth or fifth at bat, his possible home run will drive in three additional runs. How many batting orders can you form?

	Batter 1	Batter 2	Batter 3	Batter 4	Batter 5	Batter 6	Batter 7	Batter 8	Batter 9
# of choices	13	12	11	10	9	8	7	6	5

left to choose

Use the Fundamental Counting Principal:

$13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 259{,}459{,}200$

A **Permutation** is an arrangement of items in a particular order.

Permutation of *n* Things Taken *r* at a Time:

$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$

from Expl. 1 above:

${}_{13}P_9 = \frac{13!}{(13-9)!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$

4! or "4 factorial"

$\boxed{n=20}$

Example 3: You and 19 of your friends have decided to form an Internet marketing consulting firm. The group needs to choose three officers—a CEO, an operating manager, and a treasurer. In how many ways can those offices be filled?

$\boxed{r=3}$

${}_{20}P_3 = \frac{20!}{(20-3)!} = \frac{20!}{17!} = \frac{20 \cdot 19 \cdot 18 \cdot \cancel{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}$

$= \boxed{6840}$

You Try: You need to arrange seven of your favorite books along a small shelf. How many different ways can you arrange the books, assuming that the order of the books makes a difference to you?

$n=7$
$r=7$ ${}_7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

Exploration 2: Throughout the history of entertainment, performers have featured choreography in their acts. Singers who are known for their serious moves include Beyoncé, Lady Gaga, Shakira, Justin Timberlake, and Usher. Imagine that you ask your friends the following question: "Of these five entertainers, which three would you select to be included in a documentary on singers and choreography?" You are not asking your friends to rank their three favorite artists in any kind of order—they should merely select the three to be included in the documentary.

One of your friends say they would select Beyonce, Justin Timberlake, and Usher. Another friend says they would select Justin Timberlake, Usher, and Beyonce. **Are these outcomes the same or different?** same

In this situation, the order does not **matter.**

A **Combination** is an arrangement of items in which order does not matter.

Since the order does not matter in combinations, there are fewer combinations than permutations. The combinations are a "subset" of the permutations.

To find **Combinations of *n* Items chosen *r* at a Time**, you can use the formula: ${}_nC_r = \frac{n!}{(n-r)!\,r!}$

$r = 3$

Example 4: A three-person committee is needed to study ways of improving public transportation. How many committees could be formed from the eight people on the board of supervisors? $n = 8$

$${}_8C_3 = \frac{8!}{(8-3)!\,3!} = \frac{8!}{5!\,3!} = \frac{8\cdot7\cdot6\cdot\cancel{(5!)}}{\cancel{(5!)}\cdot3\cdot2\cdot1} = \boxed{56}$$

Example 5: In poker, a person is dealt 5 cards from a standard 52-card deck. The order in which you are dealt the 5 cards does not matter. How many different 5-card poker hands are possible?

$${}_{52}C_5 = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!} = \frac{52\cdot51\cdot50\cdot49\cdot48\cdot\cancel{(47!)}}{\cancel{(47!)}\cdot5\cdot4\cdot3\cdot2\cdot1} = \boxed{2{,}598{,}960}$$ Wow!

Example 6 – 9: Determine if each is a permutation or combination, then find the number of outcomes.

6. In how many ways can a President and Vice President be selected from a pool of 6 people?

Permutation

(Ranked, order matters)

$${}_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6\cdot5\cdot\cancel{4\cdot3\cdot2\cdot1}}{\cancel{4\cdot3\cdot2\cdot1}} = \boxed{30}$$

7. A political science professor must select 4 students from her class of 12 students for a field trip to state legislature. In how many ways can she do it?

Combination

(no order or rank)

$${}_{12}C_4 = \frac{12!}{(12-4)!\,4!} = \frac{12!}{8!\,4!} = \frac{12\cdot11\cdot10\cdot9\cdot\cancel{(8!)}}{\cancel{(8!)}\cdot4\cdot3\cdot2\cdot1} = \boxed{495}$$

8. The professor was asked to rank the top 4 students in her class of 12. In how many ways can that be done?

Permutation

$${}_{12}P_4 = \frac{12!}{(12-4)!} = \frac{12!}{8!} = \frac{12\cdot11\cdot10\cdot9\cancel{(8!)}}{\cancel{(8!)}} = \boxed{11880}$$

9. A police chief needs to assign officers from the 10 available to control traffic at intersections *A*, *B*, and *C*. In how many ways can he do it?

$n = 10$ $r = 3$

Combination

$${}_{10}C_3 = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = \frac{10\cdot9\cdot8\cdot\cancel{(7!)}}{\cancel{(7!)}\,3\cdot2\cdot1} = \boxed{120}$$

Example 10 – Probability: Florida's lottery game, LOTTO, is set up so that each player chooses six different numbers from 1 to 53. If the six numbers chosen match the six numbers drawn randomly, the player wins (or shares) the top cash prize. With one LOTTO ticket, what is the probability of winning this prize? It doesn't say order matters ∴

$${}_{53}C_6 = \frac{53!}{(53-6)!\,6!} = \frac{53!}{47!\,6!} = \frac{53\cdot52\cdot51\cdot50\cdot49\cdot48\cdot\cancel{(47!)}}{\cancel{(47!)}\,6\cdot5\cdot4\cdot3\cdot2\cdot1} = 22{,}957{,}480$$

Probability of winning: $\boxed{\dfrac{1}{22{,}957{,}480}}$