# 6.1 Notes: Graphing Rational Functions in Graphing Form

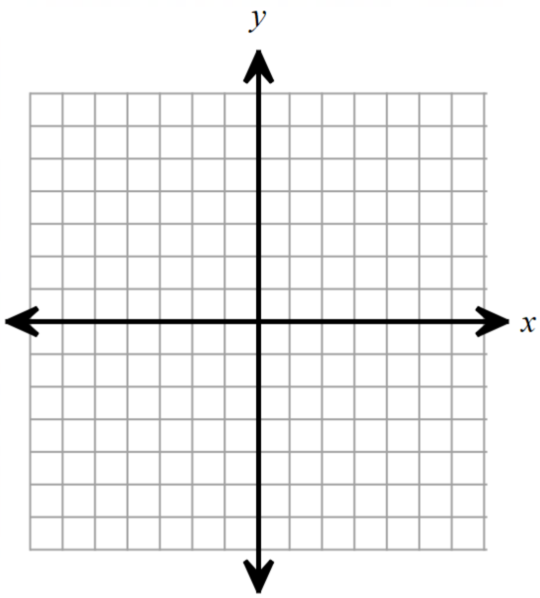
**Rational Function:**

**Asymptote:**

Note: horizontal asymptotes may be crossed (although rarely are); vertical asymptotes are never crossed.

The parent function: $y = \frac{1}{x}$

| $x$ | $y$ |
|---|---|
| 3 | |
| 2 | |
| 1 | |
| $\frac{1}{2}$ | |
| $\frac{1}{3}$ | |
| $\frac{1}{4}$ | |
| 0 | |
| $-\frac{1}{4}$ | |
| $-\frac{1}{3}$ | |
| $-\frac{1}{2}$ | |
| -1 | |
| -2 | |
| -3 | |




Vertical Asymptote:

Horizontal Asymptote:

Domain:

Range:

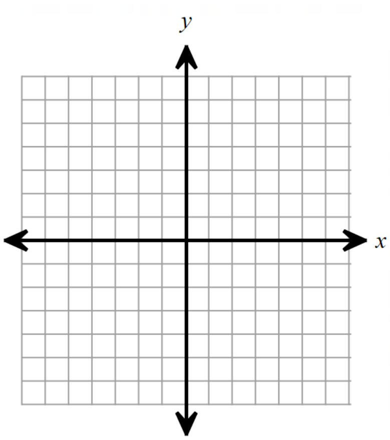
## Graphing Form of a Rational Function: $y = \frac{a}{x-h} + k$

**Horizontal Shifts:**

**Vertical Shifts:**

**Examples:** Draw a sketch of each rational function by using shifts of the parent function. Then write the equations for any asymptotes, and find the domain and range (in set notation).

1) $y = \frac{1}{x+3} - 2$



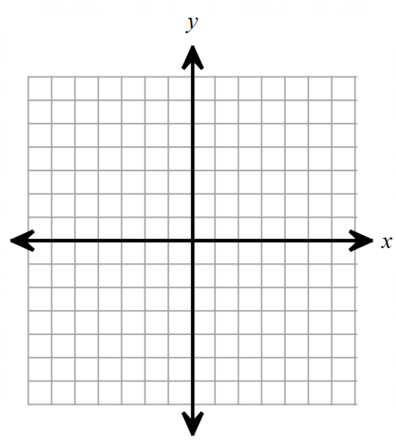

VA:

HA:

Domain:

Range:

2) $y = \frac{1}{x} + 3$



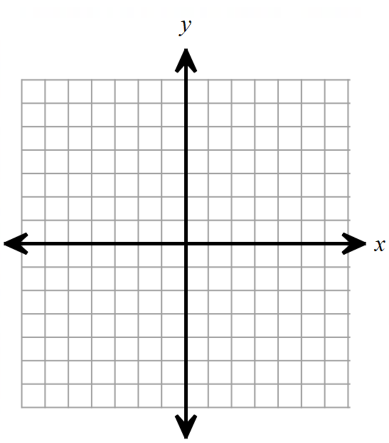

VA:

HA:

Domain:

Range:

3) $y = \frac{1}{x-4}$



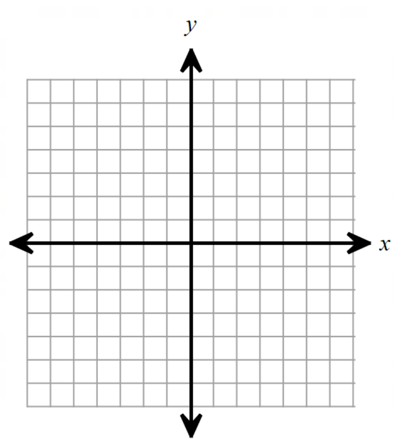

VA:

HA:

Domain:

Range:

4) $y = \frac{1}{x+2} - 5$




VA:

HA:

Domain:

Range:

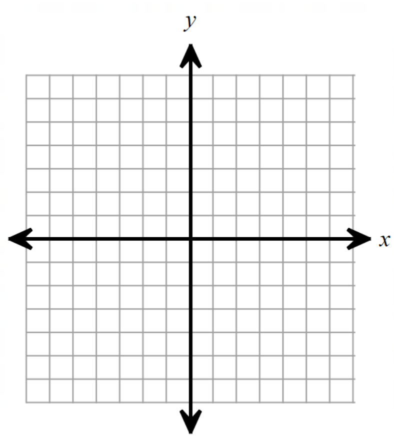
## Graphing Form of a Rational Function: $y = \frac{a}{x-h} + k$

**Vertical Reflection:**

**Vertical Stretch:**

**Examples:** Draw a sketch of each rational function by using transformations of the parent function. Then write the equations for all asymptotes, and find the domain and range (in set notation.)

5) $y = \frac{-1}{x}$



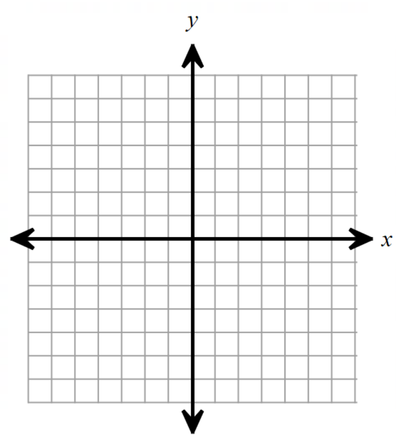

VA:

HA:

Domain:

Range:

6) $y = \frac{2}{x} + 1$



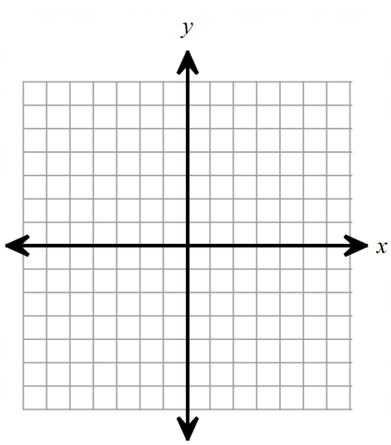

VA:

HA:

Domain:

Range:

7) $y = \frac{-1}{x+3}$



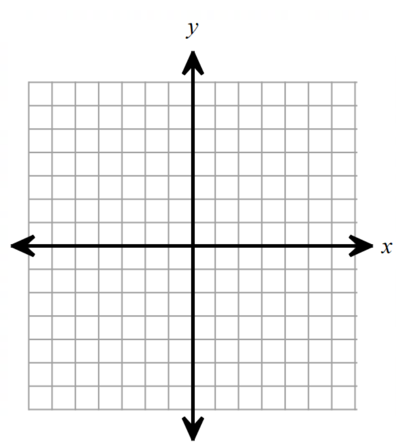

VA:

HA:

Domain:

Range:

8) $y = \frac{-3}{x-4} + 2$




VA:

HA:

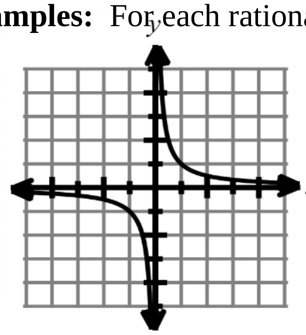
Domain:

Range:

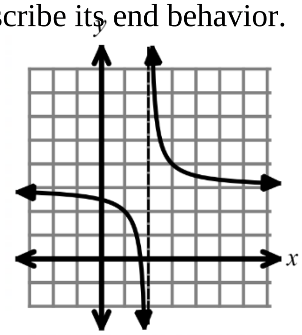
**End Behavior of a Rational Function:**

**Examples:** For each rational function, describe its end behavior.

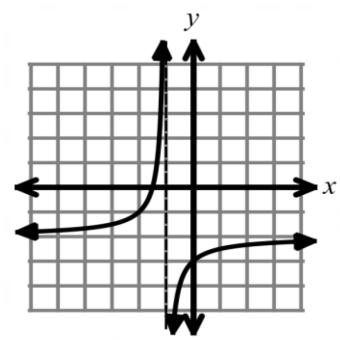
9)



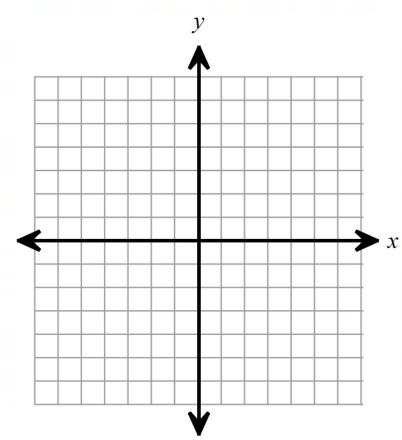

10)



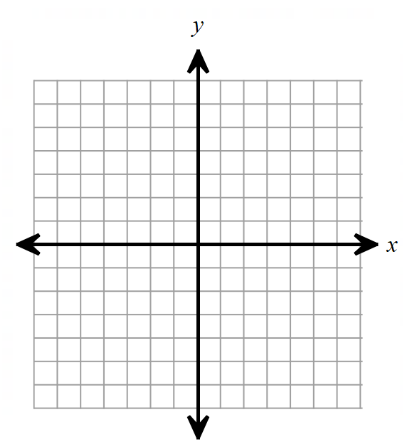

11)




12) $y = -\frac{1}{x} - 2$




13) Write an equation, in graphing form, of a rational function with a vertical asymptote at $x = 3$, a horizontal asymptote at $y = 2$, and a vertical reflection of the parent function. Then sketch the function below.



# 6.2 Notes: Graphing Rational Functions in General Form

## General Form of a Rational Function:

**Vertical Asymptotes:** A vertical line that a graph ______________________ but never touches or crosses.

- ✓ Comes from the undefined values on the ____________________.
- ✓ Cannot be a factor of the numerator.
- ✓ Factor the _________________________ to find the vertical asymptotes.
- ✓ Draw as a dotted line.

**Horizontal Asymptote:** A horizontal line that a graph approaches but can touch or cross. (Note: occasionally a horizontal asymptote can be crossed by a rational function. This is rare, but can happen close to vertical asymptotes.)

- ✓ Compare the degree of the numerator and the degree of the denominator.
- ✓ Which is growing faster, as *x* values approach $\pm\infty$?

**Case 1:** If degree of the ____________________________ is larger, then there is ________ horizontal asymptote.

**Case 2:** If the degree of the __________________________ is larger, then the horizontal asymptote is $y = 0$.

**Case 3:** If the degree of the numerator is the _______________ as the degree of the denominator, then use the leading coefficients y = $\frac{P}{Q}$.

**Examples:** For each rational function below, write the equation of any vertical and horizontal asymptotes.

1) $y = \frac{1}{x-2}$

2) $y = \frac{x-1}{x+3}$

3) $y = \frac{x^2+5}{x}$

4) $y = \frac{2x+5}{x^2-9}$

5) $\frac{7x+2x^2}{x+1}$

6) $\frac{4-3x+7x^3}{x^3+5x^2+4x}$

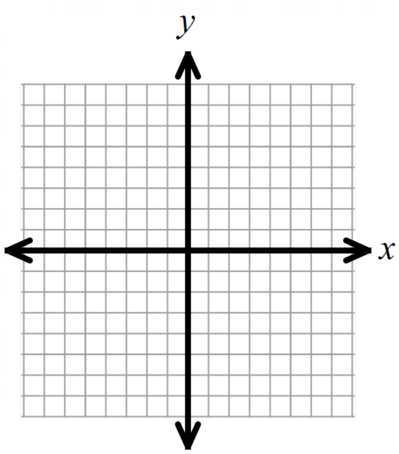
## Graphing with horizontal and vertical asymptotes and testing points:

Steps:

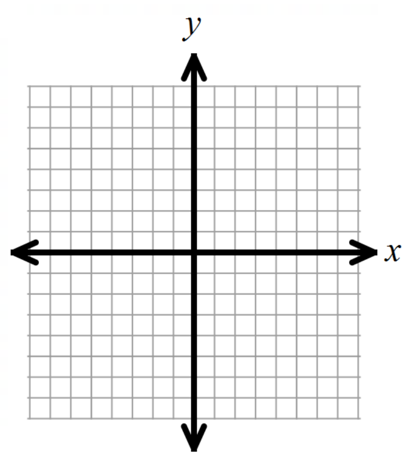
1. Factor the expression fully.
2. Find the zeros of the denominator identify all VA (reminder: those factors cannot be repeated on the numerator)
3. Compare the degree of the numerator and denominator to identify any HA.
4. Sketch the asymptotes.
5. Choose an input value on *both sides* of any VA. Find the output, and sketch at least one point on each side of each VA. Fit the curve to the asymptotes.

**Examples:** Sketch each rational function. Include all asymptotes, and identify the domain and range.

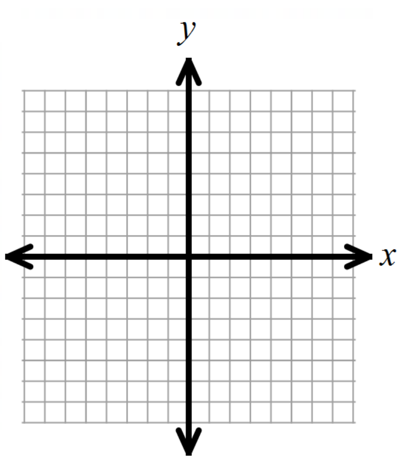
1) $y = \frac{-3}{x-2}$



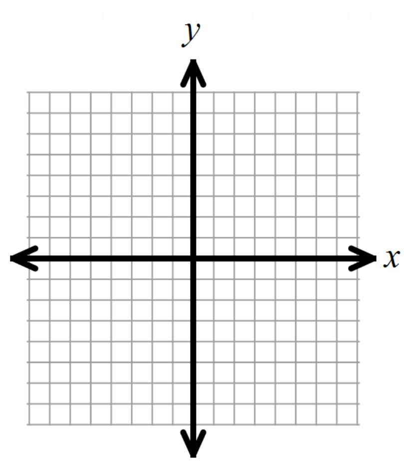

2) $f(x) = \frac{5}{x^2+2x+1}$




3) $h(x) = \frac{2x^2}{x^2-9}$




4) $y = \frac{5x+1}{x^2-x-2}$



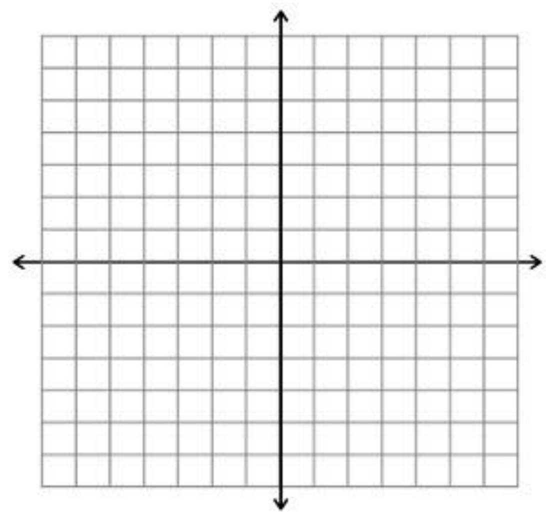
## Other helpful points when graphing rational functions:

- *y*-intercept

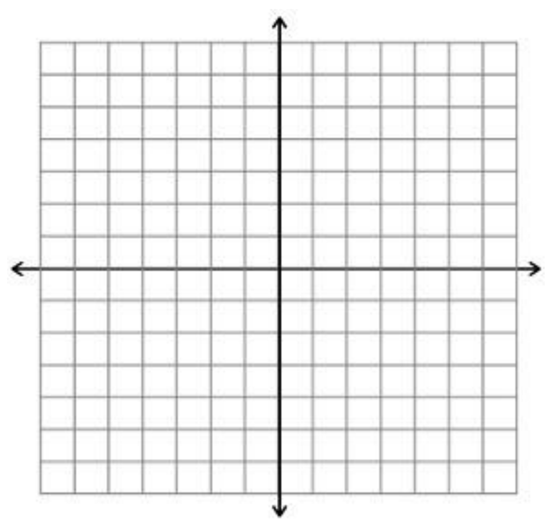
- *x*-intercepts

**Examples: Graph each rational function. Include the following: all asymptotes, all intercepts, domain, range, and end behavior.**

5) $y = \frac{x^2+5x-6}{x^2-4}$



6) $y = \frac{4x-8}{x+3}$



**Example 7:** The rational function $f(x) = \frac{1}{x}$ is transformed by a stretch by a factor of 4, a horizontal shift to the right 3 units, and a vertical shift down 2 units. Write an expression for the function after the transformations.

**Example 8:** Use synthetic division to divide $(4x - 2)$ by $(x - 1)$.

**Consider the rational expression** $y = \frac{4x-2}{x-1}$. Use your results from the synthetic division above to re-write this expression in graphing form.

## 6.3 Notes: Graphing Rational Functions with Holes

Exploration: Consider the function $g(x) = \frac{x-3}{x^2-5x+6}$

Based on what you know so far about rational functions, what do you anticipate $g(x)$ would look like? Consider asymptotes as part of your decision.

Use a graphing calculator to sketch $g(x)$. Does this look like you thought it would? What is surprising about this graph?

Find $g(3)$ by evaluating $g(x)$ at 3. What do you notice? However, what does $g(3)$ appear to be on the graph? What is happening? (Note: reduce $g(x)$ by dividing out the repeated factor, and then graph the new expression.)

**Equivalent Expressions:** Some rational expressions can be written as simpler equivalent expressions. This happens when a factor is repeated on the numerator and the denominator. Consider $f(x) = \frac{x^2-16}{x+4}$.
Factor fully, and *reduce out* any factors that repeat on the numerator and denominator.

An expression equivalent to $f(x) = \frac{x^2-16}{x+4}$ is $g(x) =$________________. Compare the graph of each by using your graphing calculator.

*Find $f(-4)$ and $g(-4)$. What do you notice?

**Holes of a rational function:** A single point that is undefined for the graph.

- ✓ Plot as an ____________________ circle.
- ✓ Is a repeated _________________ on the numerator and denominator.
- ✓ Write as an ordered pair.

**Examples: Find the *coordinates* of any holes in the rational functions below.**

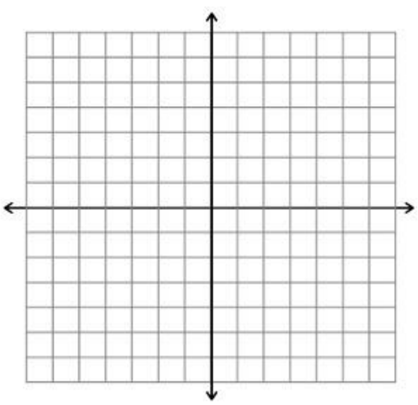
1) $y = \frac{x^2-4x}{x^2-6x+8}$ 2) $y = \frac{x^2-3x}{5x-2x^2}$ 3) $y = \frac{x+1}{x^2+7x+6}$

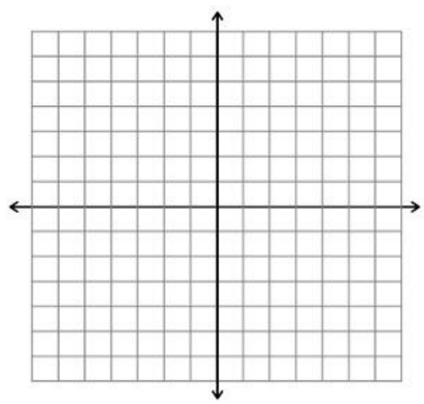
**Graphing Rational Functions with Holes**

**Examples: Graph each rational function. Include any asymptotes and holes.**

4) $y = \frac{x+5}{x^2+3x-10}$

5) $y = \frac{2x^2-6x}{x-3}$





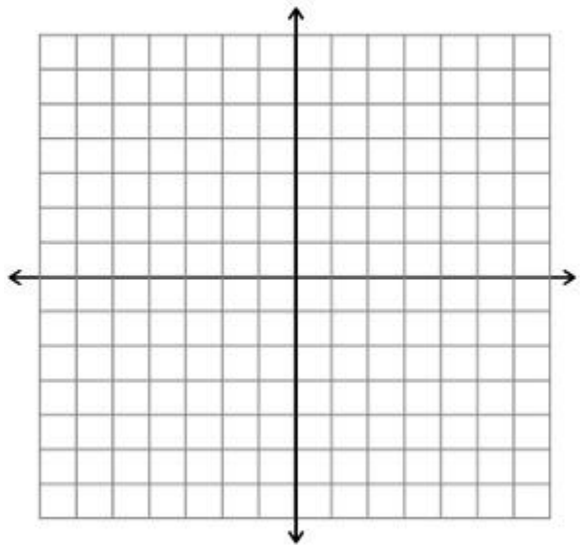
6) Find the domain and range for #5 above.

**Domain: All defined values of the input (*x*) of a function. The domain is restricted at the following values:**

**Range: All defined values of the output (*y*) of a function. The range is restricted at the following values:**

**Example 7: Graph the given rational function. Include all asymptotes, holes, and intercepts. Find the domain, range, and end behavior.**

$$y = \frac{x^2+9x+20}{x^2+7x+12}$$



## Summary for Graphing Rational Functions that are not in graphing form:

1. Factor fully.
2. Factors that reduce out identify the *x*-coordinate of any holes in the graph. Find the *y* value(s) by evaluating the reduced expression at the given *x*-value. Holes *must* be written as ordered pairs.
3. Factors of the denominator that are *not* repeated on the numerator give the values of vertical asymptotes. Write as $x$ = constant.
4. Compare the degree of the numerator and denominator to identify any horizontal asymptotes. Write as $y =$ constant.
5. Factors of the numerator that are not reduced out are the values of *x*-intercepts.
6. To find the *y*-intercept, evaluate the expression at $x = 0$. As needed, use the reduced equivalent expression.
7. As needed, test points on either side of any vertical asymptotes in order to find points on the graph. Fit the curve to the asymptotes.

# 6.4 Notes: Simplifying, Multiplying, and Dividing Rational Expressions

**Rational Function** – A function of the form $f(x) = \frac{p(x)}{q(x)}$   Ex: $y = \frac{x+2}{x^2+5x-8}$ or $y = \frac{5}{x-10}$

Domains of fractions are undefined when the denominator = 0.

**Find the domain:**

1) $y = \frac{x-2}{3x}$   2) $f(x) = \frac{7}{(x+2)(3x-5)}$   3) $g(x) = \frac{5x}{4x^3-9x}$

Reminder: What are these domain restrictions representing on the graph of the rational function?

## Simplified form of a rational expression – Factor, then divide common factors

**Examples:** Simplify completely. Identify any restrictions on the domain.

1) $\frac{x^2+7x+10}{x^2-4}$   2) $\frac{x^2+5x+4}{x^2+x-12}$

**Multiplying and Dividing Rational Expressions**

1. Factor, if possible.
2. For division, multiply by the reciprocal of the fraction after the ÷ sign.
3. Reduce repeated factors.

**Examples:** Simplify completely. Identify any restrictions on the domain.

1) $\frac{5}{4x^2} \bullet \frac{2x^3}{18} \bullet \frac{9}{15x^4}$   2) $\frac{x^2+4x-12}{x^2+11x+30} \bullet \frac{x^2-2x-35}{x+4}$

3) $\frac{21a^3}{12} \div \frac{14a}{16}$

4) $\frac{x^2 - 25}{x^2 + 2x - 3} \div \frac{x+5}{x^2 - 3x - 18}$

5) $\frac{-2x^2}{x^3 - 27} \cdot (x^2 + 3x + 9)$

6) $\frac{3x^2 + x - 2}{x^4 + 1x} \div \frac{9x^2 - 4}{5x}$

## Simplifying Complex Fractions

**Examples:** Simplify each complex fraction.

7) $\dfrac{\frac{b^2-4}{b^2-2b+1}}{\frac{b+2}{b-1}}$

8) $\dfrac{x-5}{2x^2-5x-3} \div \dfrac{\frac{x^2-4x-5}{x^2-9}}{\frac{x^2+2x+1}{x^2+5x+6}}$

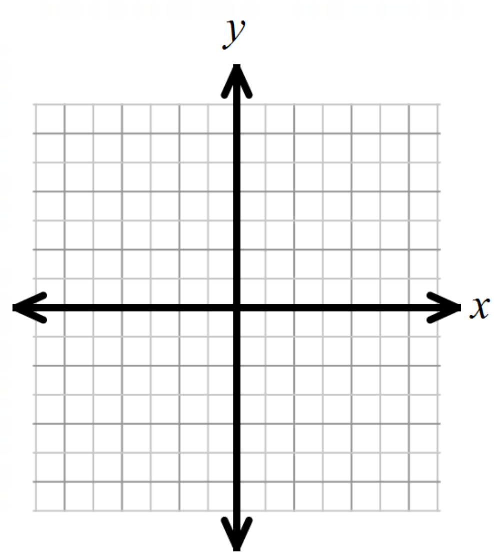
## 6.6 Notes: Solving Rational Equations & Graphing Review

**Graphing Form:** $\boldsymbol{y = \frac{a}{x-h} + k}$

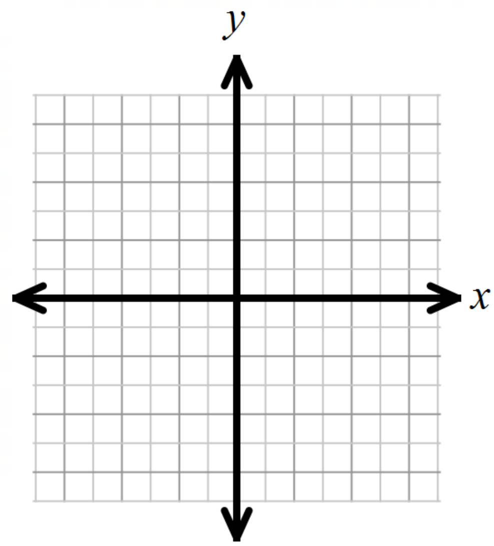
6) Graph $y = -\frac{3}{x-2} + 4$

**VA:**

**HA:**



7) Graph: $y = \frac{3x-9}{x^2-1}$, and find the requested information.

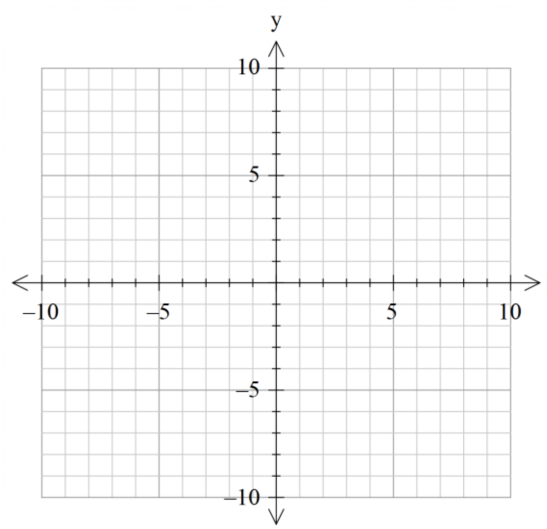



**VA:**

**HA:**

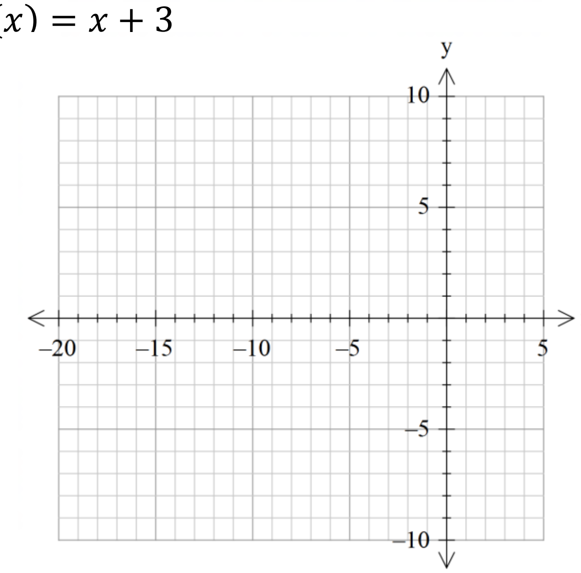
**Systems of Linear and Rational Equations**

**Examples:** Solve the following systems by using algebra and your graphing calculator.




8) Find $x$ if $f(x) = g(x)$, where $f(x) = \frac{x+2}{x-1}$ and $g(x) = 2x$.

9) Find $x$ if $h(x) = j(x)$, where $h(x) = \frac{5}{x+10} - 3$ and $j(x) = x + 3$



# 6.7 Notes: Slant (Oblique) Asymptotes

When the degree of the numerator is exactly one more than the degree of the denominator, the graph of the rational function will have a slant asymptote. *A rational function will never have more than one slant asymptote; it will also never have a horizontal asymptote and a slant asymptote at the same time.*

To find the equation of a slant asymptote, perform long division (synthetic if the denominator is a binomial of degree 1) by dividing the denominator into the numerator. As x gets very large (this is the far left or far right ends of the graph), the remainder portion becomes very small, almost zero. So, to find the equation of the slant asymptote, perform the division and discard the remainder.

***Example:*** *Find the equation of the slant asymptotes for* $y = \frac{x^2-6x+5}{x-4}, if\ any.$ *Since the degree of the numerator (2) is exactly one more than the degree of the denominator (1), a slant asymptote exists.*

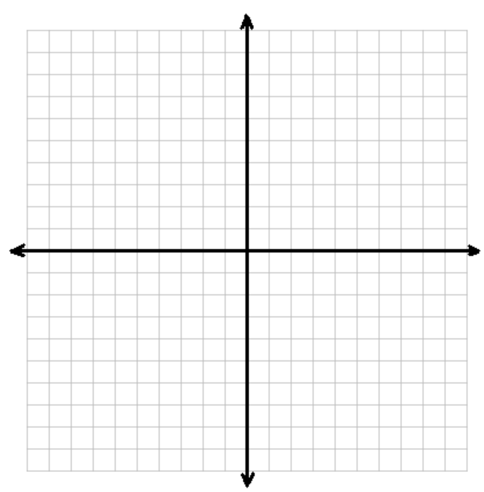
$$\begin{array}{r|rrr} 4 & 1 & -6 & 5 \\ & & 4 & -8 \\ \hline & 1 & -2 & \boxed{-3} \end{array}$$

*Disregarding the remainder, the quotient is x – 2, so the equation of the slant asymptote is y = x – 2. Note that there is also a vertical asymptote at x=4.*

**Slant Asymptotes:** Written in the form ____________________.

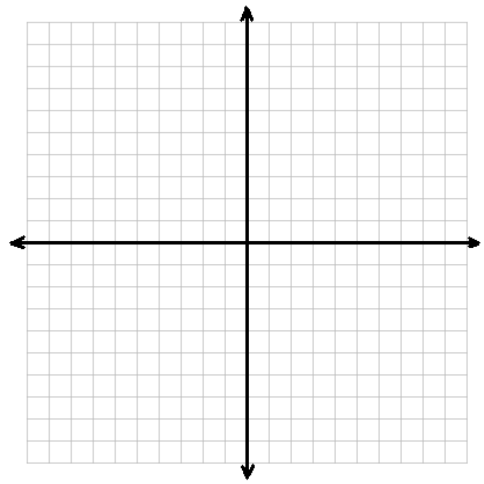
- ✓ There is a slant asymptote when the degree of the numerator is one degree higher than the ______________________.
- ✓ Use synthetic division to find the equation
  - o Ignore the remainder
- ✓ There may be a slant asymptote when there is no ________________________.

**Example 1*:*** Find the value of any slant asymptote for $y = \frac{x^2-3x-4}{x+2}$. Then graph the function and its oblique asymptote. Find the listed key features of the graph.



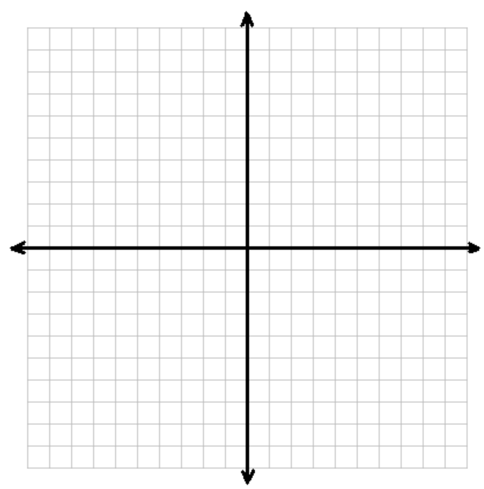
VA:
HA:
Slant Asymptote:
D:
R:
End Behavior

**Example 2:** Find the value of any slant asymptote for $y = \frac{2x^2-5}{x-1}$. Then graph the function and its oblique asymptote.



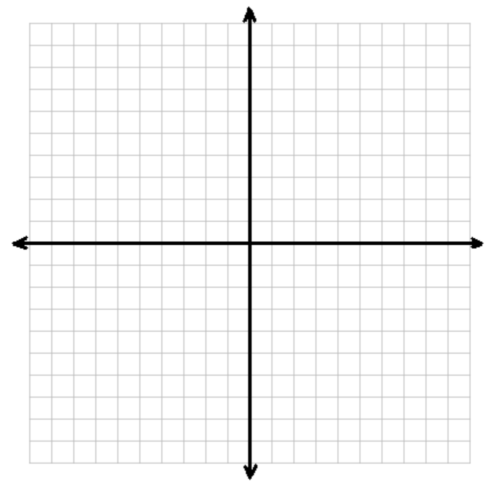
VA:
HA:
Slant Asymptote:
D:
R:
End Behavior:

**Example 3:** Graph *g(x)* = $\frac{x^2+9x+20}{x^2+7x+12}$



VA:
HA:
Slant Asymptote:
Hole:
D:
R:
End Behavior:

**Example 4:** Graph $y = \frac{x^2+4x+3}{x+1}$



VA:
HA:
Slant Asymptote:
Hole:
D:
R:
End Behavior:

## <u>Rewrite a Rational Function in Graphing Form</u>: $\boldsymbol{y = \frac{a}{x-h} + k}$

**Example 5:** What could you do to write the function, $y = \frac{2x+3}{x-1}$, in graphing form?

**Example 6:** Translate the graph of $f(x) = \frac{4x+9}{x+3}$ one unit up and two units to the right. Write the function in graphing form.