Matrix Notes Filled Out

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Matrix Unit Lesson 1: Operations with Matrices

Definition of a matrix: In mathematics, a matrix is a rectangular array (or table) of numbers, symbols, or expressions, arranged in rows and columns. Matrices help organize information.

- Matrices are "sized" using the number of rows (m) by number of columns (n).
  - Matrix A below has the following dimensions: 2 x 3
  - Matrix B below has the following dimensions: 3 x 2

2 rows, 3 columns

$$A = \begin{bmatrix} 6 & -1 & 0 \\ -2 & 4 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 11 & x \\ 3 & 7 \\ 0 & -5 \end{bmatrix}$$

3 rows 2 columns

Equal matrices have the same dimension, and corresponding elements are equal

Example 1: Solve for the values of x and y.

$$\begin{bmatrix} 12 & 0 \\ 2x & 10 \end{bmatrix} = \begin{bmatrix} 12 & y \\ 14 & 10 \end{bmatrix}$$

$\boxed{y=0}$

$2x = 14$

$\boxed{x=7}$

You Try: Find a and z.

$$\begin{bmatrix} 4 & 3 \\ -1 & 0 \\ 2z & 2.5 \end{bmatrix} = \begin{bmatrix} 6a-11 & 3 \\ -1 & 0 \\ 10 & 2.5 \end{bmatrix}$$

$6a - 11 = 4$

$6a = 15$

$\boxed{a = \frac{5}{2}}$

$2z = 10$

$\boxed{z=5}$

Scalar Multiplication is the multiplication of each element in a matrix by a single real number called a **scalar**. Basically, multiply each element by the same constant.

Example 2: Simplify: $-3\begin{bmatrix} 5 & 2 \\ -8 & 4x \end{bmatrix}$

2×2

$$\begin{bmatrix} -15 & -6 \\ 24 & -12x \end{bmatrix}$$

2×2

You Try: Cool Threads, a clothing store, uses a matrix C to represent the prices of women's clothes. The columns represent the brands Vintage, Casual, and Distressed, and the rows represent jeans and jackets. The sales tax rate is 5%. Write a matrix to represent the sales tax for each item.

$$C = \begin{array}{c} \begin{array}{ccc} \text{Vintage} & \text{Casual} & \text{Distressed} \end{array} & \\ \begin{bmatrix} 320 & 210 & 160 \\ 240 & 110 & 65 \end{bmatrix} & \begin{array}{l} \text{Jeans} \\ \text{Jackets} \end{array} \end{array}$$

$$.05\,C = \begin{bmatrix} 16 & 10.5 & 8 \\ 12 & 5.5 & 3.25 \end{bmatrix}$$

Matrix Addition and Subtraction: Matrices can only be added or subtracted if they are the same size (the same dimension).

Example 3: Determine if the matrices can be combined using addition or subtraction. If so, perform the indicated operation.

$$\underset{2\times 3}{P = \begin{bmatrix} 0 & 2 & 4 \\ 9 & 8 & 2 \end{bmatrix}} \qquad \underset{2\times 3}{Q = \begin{bmatrix} -2 & -4 & 1 \\ 9 & 7 & 0 \end{bmatrix}} \qquad \underset{3\times 3}{R = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 3 & 5 \\ 0 & -6 & 1 \end{bmatrix}}$$

✓ **a.** Q – P =

$$\begin{bmatrix} -2 & -6 & -3 \\ 0 & -1 & -2 \end{bmatrix}$$

b. P + R = not possible

<u>You Try</u>: Use the following matrices to answer the questions below.

$$A = \begin{bmatrix} 5 & -7 & 3 \\ 4 & 8 & -2 \end{bmatrix}, B = \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 3 & -4 \end{bmatrix}, C = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 15 & -9 \end{bmatrix} \quad D = \begin{bmatrix} -5 & 7 & -3 \\ -4 & -8 & 2 \end{bmatrix}$$

a. Which matrices can be combined using addition and subtraction?

A, C, D

b. Find $C + D$.

$$C + D = \begin{bmatrix} 7 & 7 & -3 \\ -4 & 7 & -7 \end{bmatrix}$$

Note: When the sum of two matrices is the zero matrix, the matrices are **additive inverses.**

c. What is the additive inverse of matrix B?

$$B + ? = 0 \qquad \begin{bmatrix} -6 & -5 \\ 2 & 0 \\ -3 & 4 \end{bmatrix}$$

Matrix Multiplication

Matrices can only be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. Reminder: matrix dimensions are written as [rows x columns]

Samples of dimensions of matrices that CAN be multiplied together.

$$\underbrace{2\times \underbrace{3 \quad 3}_{\text{same!}}\times 5}_{\text{result}} \qquad \underbrace{1\times \underbrace{4 \quad 4}_{\text{same!}}\times 2}_{\text{result}}$$

$$3\times \underbrace{5 \quad 2}_{}\times 3$$

cannot multiply

Sample: As a class, find AB if $A = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -1 \\ 0 & 6 \\ 4 & -2 \end{bmatrix}$. Then describe the process in your own words.

Dimensions check: $\underbrace{1 \times 3 \quad 3 \times 2}$ ✓

$3(5) + 1(0) + 2(4)$
$15 + 8$

$$\begin{bmatrix} 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 0 & 6 \\ 4 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 23 & -1 \end{bmatrix}}$$

$3(-1) + 1(6) + 2(-2)$
$-3 + 6 - 4$
$3 - 4$
-1

For Examples 4 – 6: Use the following matrices to find each product, if possible.

$M = \begin{bmatrix} 2 & 1 & 5 \\ 4 & -3 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 10 \\ -2 \\ 7 \end{bmatrix}$, $P = \begin{bmatrix} 8 & 10 \\ -3 & 2 \\ 1 & -5 \end{bmatrix}$, $Q = \begin{bmatrix} 7 & 5 \\ -2 & 3 \end{bmatrix}$, $R = \begin{bmatrix} -6 & 0 & 3 \end{bmatrix}$

$56 + (-20)$

4) Find MN.

$\underbrace{2 \times 3 \quad 3 \times 1}$

$$\boxed{\begin{bmatrix} 53 \\ 53 \end{bmatrix}}$$

$20 - 2 + 35$
$20 + 33$

$40 + 6 + 7$
$40 + 13$

5) Find PQ.

$\underbrace{3 \times 2 \quad 2 \times 2}$

$$\begin{bmatrix} 36 & 70 \\ -25 & -9 \\ 17 & -10 \end{bmatrix}$$

6) Find NP.

$3 \times \cancel{1} \; \cancel{3} \times 2$

not possible

Multiplicative Identity

3x3 3x3

Example 7: Find IQ, if $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -3 & 2 \\ -4 & 5 & -6 \\ 9 & -7 & 8 \end{bmatrix}$

The Matrix I is called an **Identity Matrix** because IQ = Q for every 3x3 matrix Q.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Zero Matrix

Example 8: Find CD, if $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\underbrace{2 \times 2 \quad 2 \times 1}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Zero Matrix is the Additive Identity

$A + Z = A$

Matrix Unit Lesson 2: Vectors

Vocabulary

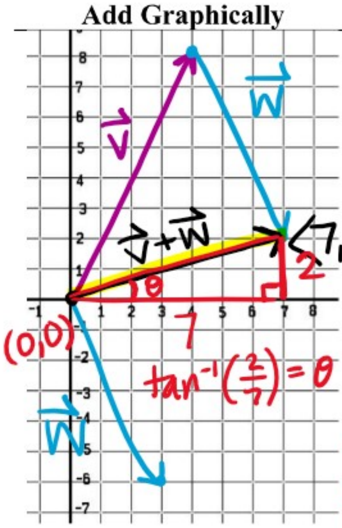
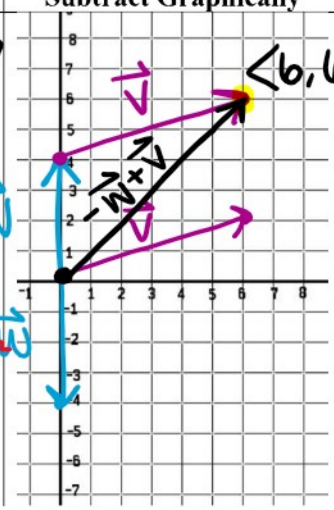
A **vector**, written as $\vec{a}$, is a quantity with both direction and magnitude.

The **direction** of a vector is considered from the initial point to the terminal point. The **magnitude** is the length of the vector written as $|\vec{a}|$.

The **component form** of a vector is represented by the coordinates $\langle x, y\rangle$ which describe the horizontal and vertical change of position from the initial to the terminal point.

Example 2: Add vectors $\vec{v} = \langle 4, 8\rangle$ and $\vec{w} = \langle 3, -6\rangle$ graphically and algebraically.

Example 3: Subtract vectors $\vec{v} = \langle 6, 2\rangle$ and $\vec{w} = \langle 0, -4\rangle$ graphically and algebraically.

Add Graphically	Add Algebraically	Subtract Graphically	Subtract Algebraically
 	$\langle 4, 8\rangle + \langle 3, -6\rangle$ $\langle 7, 2\rangle$ (0,0) (7,2) $d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$ $d = \sqrt{(2)^2 + (7)^2}$ $d = \sqrt{4+49} = \sqrt{53}$ $\lvert\vec{v}+\vec{w}\rvert = \sqrt{53}$	 	$\langle 6, 2\rangle - \langle 0, -4\rangle$ $\boxed{\langle 6, 6\rangle}$ $\vec{v} - \vec{w}$ $\vec{v} + (-\vec{w})$

You Try: Perform the following operation given $\vec{v} = \langle -3, 4\rangle$, $\vec{w} = \langle 5, -8\rangle$, $\overrightarrow{MN} = \langle 9, 12\rangle$, and $\overrightarrow{NO} = \langle 2, 7\rangle$.

a. $\vec{v} + \vec{w}$

$\langle 2, -4\rangle$

b. $\overrightarrow{MN} + \overrightarrow{NO}$

$\langle 11, 19\rangle$

c. $\vec{v} - \overrightarrow{MN}$

$\langle -12, -8\rangle$

Example 4: Multiply each vector by the given scalar. What is the magnitude and direction of the new vector? How does this compare to the original vector?

a. $\vec{r} = \langle 3, 1\rangle\,;\, scalar = 3$ (0,0) & (9,3)

$3 \cdot \vec{r} = \langle 9, 3 \rangle$

same direction
3 times as long

$d = \sqrt{(9)^2 + (3)^2} = \sqrt{81+9} = \sqrt{90}$

b. $\vec{s} = \langle 4, 3\rangle\,;\, scalar = -3$

$-3 \cdot \vec{s} = \langle -12, -9 \rangle$

3 times as long
opposite direction.

Use Matrices to Transform a Vector

Example 4: Rotate $\overrightarrow{AB} = \langle 7, 9\rangle$ 180° around the origin using matrices.
Recall that a 180° rotation about the origin is represented by the rule: $(x,y) \to (-x,-y)$
Write this rule as a matrix: $\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 7 \\ 9 \end{bmatrix}}_{2\times 1} = \begin{bmatrix} -7 \\ -9 \end{bmatrix}$

or

$\langle -7, -9 \rangle$

Example 5: Reflect $\overrightarrow{CD} = \langle 6, 2\rangle$ across the x-axis using matrices.
Recall that a reflection over the x-axis is represented by the rule: $(x,y) \to (x,-y)$
Write this rule as a matrix: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

or

$\langle 6, -2 \rangle$

You Try: Let $\overrightarrow{EF} = \langle 5, 10\rangle$. How is $\overrightarrow{EF}$ transformed when it is multiplied by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$ or $\langle -5, 10 \rangle$

reflection over y-axis

Example 6: A segment with endpoints $D(-4, 5)$ and $E(-1, 7)$ can be represented by the matrix $\begin{bmatrix} -4 & -1 \\ 5 & 7 \end{bmatrix}$. $\overline{DE}$ is translated using the matrix operation $\begin{bmatrix} -4 & -1 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix}$. How is $\overline{DE}$ translated?

$\begin{bmatrix} -7 & -4 \\ 3 & 5 \end{bmatrix}$

left 3 units
down 2 units

Matrix Unit Lesson 3: Inverses, Determinants, and Solving Systems

The **Determinant** of a 2 x 2 matrix A, denoted *det A*, is the value $ad - bc$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Example 1: Find the determinant of the following matrices.

a. $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$ $8 - -1 = \boxed{9}$

b. $\begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}$ $-12 - -12$ $\boxed{0}$

c. $\begin{bmatrix} 2 & 2 & 1 \\ 0 & -1 & 0 \\ 3 & 0 & -4 \end{bmatrix}\begin{matrix} 2 & 2 \\ 0 & -1 \\ 3 & 0 \end{matrix}$

$-3 + 0 + 0$ $\boxed{-3}$ top

$8 + 0 + 0$ $\boxed{8}$ bottom

bottom - top

$8 + +3$

$\boxed{11}$

An **Inverse Matrix** of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix $\begin{bmatrix} w & x \\ y & z \end{bmatrix}$ such that the product of the two matrices is the identity matrix.

~~**Example 2**~~: What is the inverse matrix of $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$?

$5 \cdot \frac{1}{5} = 1$

$[A] \cdot [B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$[B] \cdot [A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

inverse

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and A has an inverse, then $A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The inverse exists if and only if det ≠ 0

Example 3: Does each given matrix have an inverse? If so, find it. If the dimensions are 3x3, only find the determinant.

a. $\begin{bmatrix} -4 & -2 \\ -6 & 3 \end{bmatrix}$ det = -12 - 12 = -24

$\frac{1}{-24}\begin{bmatrix} 3 & 2 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} \frac{3}{-24} & \frac{2}{-24} \\ \frac{6}{-24} & \frac{-4}{-24} \end{bmatrix}$

$\boxed{\begin{bmatrix} -1/8 & -1/12 \\ -1/4 & 1/6 \end{bmatrix}}$

b. $\begin{bmatrix} 7 & 2 \\ 14 & 4 \end{bmatrix}$

det = 28 - 28 = 0

Inverse does not exist since det = 0

c. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 4 \end{bmatrix}$

You Try: Does each given matrix have an inverse? If so, find it. If the dimensions are 3x3, only find the determinant.

* graphing calculator for inverse

a. $\begin{bmatrix} 10 & 2 \\ -5 & 3 \end{bmatrix}$ det = 30 − −10 = 40

b. $\begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$

$[A]^{-1}$

$$\frac{1}{40}\begin{bmatrix} 3 & -2 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} \frac{3}{40} & \frac{-2}{40} \\ \frac{5}{40} & \frac{10}{40} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3}{40} & \frac{-1}{20} \\ \frac{1}{8} & \frac{1}{4} \end{bmatrix}}$$

Inverse Matrices and Systems of Equations

Example 6: Write each system of linear equations as a matrix equation, then solve the system. Note: Use a graphing calculator for 3 variable systems.

a. $\begin{cases} 10x - 9y = 1 \\ 7x + 6y = 13 \end{cases}$

b. $\begin{cases} 4x + 2y - z = 14 \\ 2x - 3y + 5z = 20 \\ 3x - 6y = 8 \end{cases}$

graphing calculator

$$A^{-1}\cdot\cancel{\underbrace{\begin{bmatrix} 10 & -9 \\ 7 & 6 \end{bmatrix}}_{\text{Inverse}}}\cdot\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\cdot\begin{bmatrix} 1 \\ 13 \end{bmatrix}$$

det = 60 − −63 = 123

$$\frac{1}{123}\begin{bmatrix} 6 & 9 \\ -7 & 10 \end{bmatrix} = \begin{bmatrix} \frac{6}{123} & \frac{9}{123} \\ \frac{-7}{123} & \frac{10}{123} \end{bmatrix}$$

$$\frac{6}{123} + \frac{117}{123} = \frac{123}{123} = 1$$

$$\cancel{\begin{bmatrix} \frac{6}{123} & \frac{9}{123} \\ \frac{-7}{123} & \frac{10}{123} \end{bmatrix}\begin{bmatrix} 10 & -9 \\ 7 & 6 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{6}{123} & \frac{9}{123} \\ \frac{-7}{123} & \frac{10}{123} \end{bmatrix}}_{2\times 2}\cdot\underbrace{\begin{bmatrix} 1 \\ 13 \end{bmatrix}}_{2\times 1}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{-7}{123} + \frac{130}{123} = \frac{123}{123} = 1$$

c. $\begin{cases} -x + 2y = 8 \\ -3x + 6y = -12 \end{cases}$

d. $\begin{cases} 9x + 2y + 3z = 1 \\ -8x - 3y - 4z = 1 \\ 12x + y - 2z = -17 \end{cases}$

<u>Example 7</u>: A company makes men's and women's sneakers. For the women's sneakers, the cost of materials is \$12 and labor is \$10. For the men's sneakers, the materials cost \$18 and the labor costs \$14. Last week, the company spent \$340 on labor and \$420 on materials. How many sneakers of each type did the company produce?

Let w =

Let m =

You Try - Modeling: Steve wants to mix three different types of cereal to create a mixture with 3,400 calories, 90 grams of protein, and 90 grams of fiber. The boxes of cereal show the number of calories, grams of protein, and grams of fiber in one serving of cereal A, B, and C. Write a matrix equation to represent the situation. How many servings of each type of cereal does Steve need to include in the mixture?