10.1 Notes: Sequences and Summation Notation

Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …
(add 2 previous terms)

An infinite series $\{a_n\}$ is a function whose domain is the set of positive integers. The function values, or terms, of the sequence are represented by
Sequences whose domains consist only of the first n positive integers are called finite sequences. finite → with end

Example 1: Write the first 4 terms of each sequence.

a) $a_n = 2n + 5$

$a_1 = 2(1) + 5 = 7$
$a_2 = 2(2) + 5 = 9$
$a_3 = 2(3) + 5 = 11$
$a_4 = 2(4) + 5 = 13$

$\boxed{7, 9, 11, 15}$

b) $a_n = \frac{(-1)^n}{2^n+1}$

$-\frac{1}{3}, \frac{1}{5}, -\frac{1}{9}, \frac{1}{17}$

$a_1 = \frac{(-1)^1}{2^1+1} = -\frac{1}{3}$ $a_3 = \frac{(-1)^3}{2^3+1} = -\frac{1}{9}$

$a_2 = \frac{(-1)^2}{2^2+1} = \frac{1}{5}$ $a_4 = \frac{(-1)^4}{2^4+1} = \frac{1}{17}$

Recursion Formula:

a_1 = 1st term
a_n = described in relation to the previous term (a_{n-1})

Example 2) Find the first 4 terms of the sequence in which $a_1 = 3$, $a_n = 2a_{n-1} + 5$, for $n \geq 2$.

$a_1 = 3$
$a_2 = 2(3) + 5 = 11$
$a_3 = 2(11) + 5 = 27$
$a_4 = 2(27) + 5 = 59$

$\boxed{3, 11, 27, 59}$

Factorial Notation:

$n! = n(n-1)(n-2)(n-3)\ldots 1$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Example 3: Write the first 4 terms of $a_n = \frac{20}{(n+1)!}$. → $\boxed{10, \frac{10}{3}, \frac{5}{6}, \frac{1}{6}}$

$a_1 = \frac{20}{2!} = \frac{20}{2 \cdot 1} = 10$
$a_2 = \frac{20}{3!} = \frac{20}{3 \cdot 2 \cdot 1} = \frac{20}{6} = \frac{10}{3}$
$a_3 = \frac{20}{4!} = \frac{20}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{20}{24} = \frac{5}{6}$
$a_4 = \frac{20}{5!} = \frac{20}{120} = \frac{1}{6}$

Example 4: Evaluate each factorial expression:

a) $\frac{14!}{2!12!} = \frac{14\cdot 13\cdot \cancel{12!}}{2\cdot 1\cdot \cancel{12!}} = 7\cdot 13 = \boxed{91}$

b) $\frac{n!}{(n-1)!}$ $\frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = \boxed{n}$

c) $\frac{(n+2)!}{n+2}$ $\frac{\cancel{(n+2)}\cdot(n+1)!}{\cancel{(n+2)}} = \boxed{(n+1)!}$

Summation Notation:

$$\sum_{i=1}^{n} a_i = \boxed{a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n}$$

"Sigma" → "sum" or "summation"

Example 5: Expand and evaluate each series.

a) $\sum_{i=1}^{6} 2i^2$ $2(1)^2 + 2(2)^2 + 2(3)^2 + 2(4)^2 + 2(5)^2 + 2(6)^2$

$= \cancel{2} + \cancel{8} + \cancel{18} + \cancel{32} + \cancel{50} + 72 = \boxed{182}$

b) $\sum_{k=3}^{5}(2^k - 3)$ $= (2^3 - 3) + (2^4 - 3) + (2^5 - 3)$

$= 5 + 13 + 29 = 18 + 29 = \boxed{47}$

c) $\sum_{i=1}^{5} 4$ $4 + 4 + 4 + 4 + 4 = \boxed{20}$

Example 6: Express each sum using summation notation:

a) $1^2 + 2^2 + 3^2 + \cdots + 9^2$

$\sum_{i=1}^{9} i^2$

b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$

$\sum_{i=1}^{n} \frac{1}{2^{n-1}}$

c) 25 + 50 + 75 + … + 250

$\sum_{i=1}^{10} 25i$

d) $a + (a + d) + (a + 2d) + \cdots + (a + nd)$

$\sum_{i=0}^{n} (a + nd)$

10.2 Notes: Arithmetic Sequences

Arithmetic sequence: terms have the same value repeatedly added or subtracted

What is a **common difference?**

$d \to$ item added/subtracted
$\to$ rate of change
$\to$ slope!

1, 4, 7, 10, 13, ... $d=3$

18, 11, 4, -3, ... $d=-7$

Examples: Write the first six terms of each arithmetic sequence.

1) $a_1 = 6$ and $a_n = a_{n-1} - 2$ $\leftarrow d=-2$

6, 4, 2, 0, -2, -4

2) $a_1 = 100$ and $a_n = a_{n-1} + 30$

100, 130, 160, 190, 220, 250

General term of an Arithmetic Sequence (also called "explicit form")

$a_n = dn + a_0$ $\boxed{\text{or}}$ $a_n = a_1 + d(n-1)$

Examples:

3) Find the ninth term of the arithmetic sequence with a first term of 6 and a common difference of -5.

$a_1 = 6$ $d=-5$

$a_0 = 6+5 = 11$

$\boxed{a_n = -5n + 11}$

$a_9 = -5(9) + 11 = -45 + 11$

$\boxed{a_9 = -44}$

OR: $a_n = 6 + -5(n-1)$

$= 6 - 5n + 5$

$\boxed{a_n = -5n + 11}$

$a_9 = -5(9) + 11 = -44$

4) Find the eighth term of the arithmetic sequence with a first term of 4 and a common difference of 7.

$a_1 = 4$ $d=7$

$a_0 = 4 - 7 = -3$

$\boxed{a_n = 7n - 3}$

$a_8 = 7(8) - 3$

$= 56 - 3$

$\boxed{a_8 = 53}$

Example 5) Teachers in the US earned an average of \$44,600 in 2002. This amount has increased by approximately \$1130 per year.

a) Write a formula for the nth term of the arithmetic sequence that describes teachers' average earnings n years after 2001.

$a_1 = 44{,}600 \quad d = 1130$

$a_0 = 43{,}300$

$\boxed{a_n = 1130d + 43300}$

b) How much will US teachers earn, on average, by the year 2025?

$n = 24$ yrs later

$a_{11} = 1130(24) + 43{,}300$

$= \boxed{70{,}420}$

Actual data

in 2023

\$68,469 in US

in 2023 → Nvada

\$58,008

Example 6) Americans are eating more meals behind the wheel. In 2004, we averaged 32 a la car meals, → per year which is increasing by approximately 0.7 meal per year.

a) Write a formula for the nth term of the arithmetic sequence that models the average number of car meals n years after 2003.

$\boxed{a_n = 0.7n + 31.3}$

$a_1 = 32 \quad d = 0.7$

$a_0 = 32 - 0.7$

$= 31.3$

b) How many car meals will Americans average by the year → 2023

2023 → 20 yrs

$a_{11} = 0.7(20) + 31.3$

$\boxed{a_{16} = 45.3 \text{ meals}}$

Sum of the First *n* Terms of an Arithmetic Sequence:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Sum of first n terms; 1st term; last term

a_1 is the first term

a_n is the nth term

Example 7) Find the sum of the first 100 terms of the arithmetic sequence: 1, 3, 5, 7, …

$a_1 = 1 \quad d = 2$

$a_0 = -1$

$a_n = 2n - 1$

$a_{100} = 2(100) - 1 = 199$

$S_{100} = \frac{100}{2}(1 + 199) = \boxed{10{,}000}$

Example 8) Find the sum of the first 15 terms of the arithmetic sequence: 3, 6, 9, 12, ….

$a_1 = 3 \quad d = 3$

$a_0 = 0$

$a_n = 3d$

$a_{15} = 3(15)$

$= 45$

$S_{15} = \frac{15}{2}(3 + 45)$

$\boxed{S_{15} = 360}$

Example 9) Find the following sum: $\sum_{i=1}^{25}(5i-9)$

$a_1 = 5(1) - 9 = -4$

$a_{25} = 5(25) - 9 = 116$

$S_{25} = \frac{25}{2}(-4+116)$

$S_{25} = \boxed{1400}$

* also show on calc
math, 0, enter terms (use x)

Example 10) Find the following sum: $\sum_{i=1}^{30}(6i-11)$

$a_1 = 6(1) - 11 = -5$

$a_{30} = 6(30) - 11 = 169$

$S_{30} = \frac{30}{2}(-5+169)$

$\boxed{S_{30} = 2460}$

Example 11) Your grandmother has assets of $500,000. One option that she is considering involves an adult residential community for a six-year period beginning in 2009. The model $a_n = 1800n + 64130$ describes yearly adult residential community costs n years after 2008.

a) Does your grandmother have enough to pay for the facility for six years?

$a_1 = 1800(1) + 64130 = 65930$

$a_6 = 1800(6) + 64130 = 74930$

$S_6 = \frac{6}{2}(65930 + 74930)$

$S_6 = \$422,580$

b) How much would it cost for the adult residential community for a ten-year period beginning in 2009?

$a_{10} = 1800(10) + 64130 = 82130$

$S_{10} = \frac{10}{2}(65930 + 82130)$

$\boxed{S_{10} = \$740,300}$

10.3 Notes: Geometric Sequences

A sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. The amount by which we multiply each time is called the common ratio (r) of the sequence.

$5, 15, 45, 135, 405, \ldots \quad r=3$

$200, 50, \frac{25}{2}, \frac{25}{8}, \frac{25}{34}, \ldots \quad r=\frac{1}{4}$

$7, -14, 28, -56, \ldots \quad r=-2$

Example 1) Write the first six terms of the geometric sequence with first term of 12 and a common ratio of $\frac{1}{2}$.

$12, 6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$

General term of a Geometric Sequence: $a_n = \underbrace{a_1}_{\text{1st term}}\underbrace{(r)}_{\text{common ratio}}{}^{n-1}$ note: $(r \neq 1)$

Examples:

2) Find the eighth term of the geometric sequence whose first term is −4 and whose common ratio is −2.

$a_1 = 4 \quad r=-2$

$a_8 = 4(-2)^{8-1} = 4(-2)^7 = \boxed{-512}$

3) Find the seventh term of the geometric sequence whose first term is 5 and whose common ratio is $\frac{1}{3}$.

$a_1 = 5 \quad r=\frac{1}{3}$

$a_7 = 5\left(\frac{1}{3}\right)^{7-1} = 5\left(\frac{1}{3}\right)^6 = \boxed{\frac{5}{729}} \approx 0.00686$

4) Write the general term for the geometric sequence 3, 6, 12, 24, 48, … Then use the formula for the general term to find the ~~eighth~~ 12th term.

$a_1 = 3 \quad r=2$

$\boxed{a_n = 3(2)^{n-1}}$

$a_8 = 3(2)^{11}$

$\boxed{a_8 = 6144}$

Sum of the First n Terms of a Geometric Sequence: $S_n = \frac{a_1(1-r^n)}{1-r}$ note: $(r \neq 1)$

a_1 → 1st term

r = common ratio

n = # of terms

For Examples 5 – 8, find the requested sum.

5) Find the sum of the first 18 terms of the geometric sequence: 2, −8, 32, −128, ….

$a_1 = 2$ $r = -4$ $n = 18$

$$S_{18} = \frac{2(1-(-4)^{18})}{1-(-4)} = \boxed{-2.749 \times 10^{10}}$$

6) Find the sum of the first nine terms of the geometric sequence: ~~2, −6, 18, −54, …~~. $300, 100, \frac{100}{3}, \frac{100}{9}, \ldots$

$a_1 = 300$ $r = \frac{1}{3}$ $n = 9$

$$S_9 = \frac{300(1-(\frac{1}{3})^9)}{[1-(\frac{1}{3})]} = \boxed{449.977}$$

7) Find the following sum: $\sum_{i=1}^{10} 6 \cdot 2^i$ → $r = 2$

$a_1 = 6 \cdot 2 = 12$

$r = 2$

$n = 10$

$$S_{10} = \frac{12(1-2^{10})}{1-2} = \boxed{12,276}$$

8) Find the following sum: $\sum_{i=1}^{8} 2 \cdot 3^i$

$a_1 = 2 \cdot 3^1 = 6$

$r = 3$

$n = 8$

$$S_8 = \frac{6(1-3^8)}{1-3} = \boxed{19,680}$$

Computing a Lifetime Salary:

9) A job pays a salary of $30,000 the first year. During the next 29 years, the salary increases by 6% each year. What is the total lifetime salary over the 30-year period?

100 + 6 = 106%

$r = 1.06$

$a_1 = 30,000$

$r = 1.06$

$n = 30$

$$S_{30} = \frac{30000(1-1.06^{30})}{1-1.06}$$

$$S_{30} = 2,371,745.586 \approx \boxed{\$2,371,745.59}$$

What is an annuity? * savings fund * contribute set amount on a regular basis

* earns interest

* can be compounded (calculated) annually, monthly, weekly daily, continu..

amount

Annuity Formula: $A = \dfrac{P\left[\left(1+\frac{r}{n}\right)^{nt}-1\right]}{\frac{r}{n}}$

more space

P = principal (# contributing) r = interest rate as a decimal n = # times compounded per year t = time in years

Example 10) At age 30, to save for retirement, you decide to deposit \$100 at the end of each month into an IRA that pays 9.5% compounded monthly. $P = 100$ $n = 12$ $r = 0.095$

a) How much will you have from the IRA when you retire at age 65? → $t = 35$

$$A = \frac{100\left[\left(1+\frac{0.095}{12}\right)^{12\cdot 35}-1\right]}{\left(\frac{0.095}{12}\right)} = \boxed{\$333{,}946.30}$$

b) Find the amount of interest earned.

deposited: \$100·12·35 = 42,000

interest = Total − amount deposited
= 333946.30 − 42,000
= $\boxed{\$291{,}946.30}$

Sum of an Infinite Geometric Series: If $-1 < r < 1$, then the sum of the infinite geometric series is given by

$$S = \frac{a_1}{1-r}$$

infinite sum

Exploration: What if $|r| \geq 1$? Would the infinite series have a sum? Explain.

No... grows w/o bound → 1+2+4+8+16+32+64+... keeps growing forever

Example 11) Find the sum of the infinite geometric series: $\frac{3}{8} - \frac{3}{16} + \frac{3}{32} - \frac{3}{64} + \cdots$

$a_1 = \frac{3}{8}$

$r = -\frac{1}{2}$ $\{ -1 < r < 1$ ✓

$$S = \frac{\frac{3}{8}}{1 - -\frac{1}{2}} = \frac{\frac{3}{8}}{\frac{3}{2}} = \frac{3}{8}\cdot\frac{2}{3} = \boxed{\frac{1}{4}}$$

Example 12) Find the sum of the infinite geometric series: $3 + 2 + \frac{4}{3} + \frac{8}{9} \ldots$

$a_1 = 3$

$r = \frac{2}{3}$ $\{ -1 < r < 1$

$$S = \frac{3}{1-\frac{2}{3}} = \frac{3}{\frac{1}{3}} = 3\cdot\frac{3}{1} = \boxed{9}$$

Example 13) Express $0.\bar{8}$ as a fraction in lowest terms.

$0.88888 = \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \frac{8}{10000} + \ldots$

$a_1 = \frac{8}{10}$

$r = \frac{1}{10}$

$$S = \frac{\frac{8}{10}}{1-\frac{1}{10}} = \frac{\frac{8}{10}}{\frac{9}{10}} = \boxed{\frac{8}{9}}$$

10.5 Notes: The Binomial Theorem

Combination: ${}_nC_r$ or $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

total #selected

Note: for non-negative integers n and r, with $n \geq r$.

note #2: $0! = 1$ ... why?

Examples 1 – 4: Evaluate each combination:

1. $\binom{6}{4}$ $\frac{6!}{4!2!} = \frac{6\cdot5\cdot\cancel{4!}}{\cancel{4!}2\cdot1} = \boxed{15}$
2. $\binom{6}{0}$ $\frac{6!}{1\cdot6!} = \boxed{1}$
3. $\binom{8}{2}$ $\frac{8!}{2!6!} = \frac{8\cdot7}{2\cdot1} = \frac{56}{2} = \boxed{28}$
4. $\binom{3}{3}$ $\frac{3!}{3!0!} = \boxed{1}$

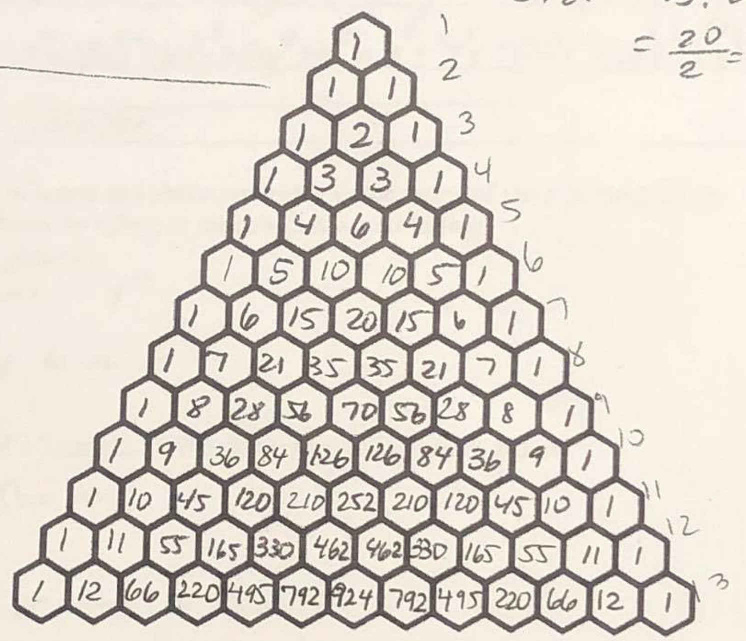
On the calculator:

n MATH r
? PROB
\# 3 nCr

${}_5C_3 = 10!$

$\frac{5!}{3!2!} = \frac{5\cdot4\cdot\cancel{3!}}{\cancel{3!}2!} = \frac{20}{2} = 10$

Pascal's Triangle:



Binomial Expansion:

multiply out a binomial taken to a power

$(x-3)^4 = (x-3)(x-3)(x-3)(x-3) \rightarrow$ mult out

Examples 1 – 3: Expand each binomial.

1) $(x+2)^2$ $x^2 + 4x + 4$

2) $(x-5)^2$ $x^2 - 10x + 25$

3) $(4m+3n)^2$ $16m^2 + 24mn + 9n^2$

Exploration: Fill in the following table. ***Hint:** to expand $(x+y)^3$, you can multiply $(x+y)^2$ by $(x+y)^1$.

Binomial	Expansion
$(x+y)^0$	1
$(x+y)^1$	$x+y$
$(x+y)^2$	$x^2+2xy+y^2$
$(x+y)^3$	$x^3+2x^2y+xy^2+x^2y+2xy^2+y^3 = x^3+3x^2y+3xy^2+y^3$
$(x+y)^4$	$\cancel{x^4}+3x^3y+\cancel{3x^2y^2}+xy^3+\cancel{x^3y}+\cancel{3x^2y^2}+3xy^3+y^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$
$(x+y)^5$	$\cancel{x^5}+\cancel{4x^4y}+6x^3y^2+4x^2y^3+y^4+\cancel{x^4y}+4x^3y^2+6x^2y^3+4xy^4+y^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$
~~$(x+y)^6$~~	remove

more room needed

- Write conjectures about the number of terms and about symmetry in the terms of the expansion in any row of the table. Verify your conjectures by filling in the row that would follow.

• ① more term than the power
• 1st term → x^n . . . last term . . . y^n
• symmetrical
• powers of each term add to n.

- Compare your expansions and Pascal's Triangle. Write down your observations below.

The coeff match the rows of Pascal's Δ

• the power $+1$ = row #

• then 2nd # match

- Use the pattern you saw to expand $(x+y)^{10}$.

use row 11.

$x^{10}+10x^9y+45x^8y^2+120x^7y^3+210x^6y^4+252x^5y^5+210x^4y^6+120x^3y^7+$
$45x^2y^8+10xy^9+y^{10}$

Examples 4 – 6: Expand the following binomials.

4) $(x+2)^4$

$1x^4 + 4x^3(2)^1 + 6x^2(2)^2 + 4\cdot x(2)^3 + 1\cdot 2^4$

$\boxed{x^4 + 8x^3 + 24x^2 + 32x + 16}$

5) $(x-2y)^5$

$1x^5 + 5x^4(-2y)^1 + 10x^3(-2y)^2 + 10x^2(-2y)^3 + 5x(-2y)^4 + 1(-2y)^5$

$\boxed{x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5}$

6) $(3x+2y)^5$

$1(3x)^5 + 5(3x)^4(2y) + 10(3x)^3(2y)^2 + 10(3x)^2(2y)^3 + 5(3x)(2y)^4 + 1(2y)^5$

$\boxed{243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240xy^4 + 32y^5}$

Binomial Coefficient: Combinations can be used to find the coefficients for each term when a binomial is expanded. These same values can also be found in Pascal's Triangle.

Finding a particular term in a Binomial Expansion:

The $(r+1)^{st}$ term of the expansion of $(a+b)^n = \binom{n}{r}a^{n-r}b^r$

Examples 7 – 8: Find the indicated term in each expansion.

7) 5^{th} term of $(2x+y)^9$

$5 = 4+1$

$r = 4$

$= \boxed{126}(2x)^5y^4$

$\binom{9}{4}$

${}_9C_4$

$\boxed{=4032x^5y^4}$

8) 7^{th} term of $(x-2)^{10}$

$7 = r+1$

$6 = r$

$\boxed{210}x^4(-2)^6 = \boxed{13440x^4}$

$\binom{10}{6}$

${}_{10}C_6$